Exam 1, Math 305, Spring 2018

Last name, first name, ID number : Score: /45

1. Suppose A and B are events having probabilities $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cup B) = 0.7$. Calculate

   (a) (2 points) $P(A \cap B) =$
   Solution $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.2$.

   (b) (2 points) $P(A \cap B') =$
   Solution $P(A \cap B') = P(A) - P(A \cap B) = 0.1$.

   (c) (2 points) $P(A|B) =$
   Solution: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$.

   (d) (2 points) Are A and B independent? Justify your answer.
   Solution: $P(A \cap B) \neq P(A)P(B)$ so A and B are not independent.

2. Suppose that $P(A|B) = 0.3$, $P(A|B') = 0.2$ and $P(B) = 0.4$. Calculate

   (a) (2points) $P(A) =$
   Solution: By the law of total probabilities, $P(A) = P(A \cap B) + P(A \cap B') = P(A|B)P(B) + P(A|B')P(B') = 0.24$.

   (b) (2points) $P(B|A) =$
   Solution: By Bayes' Rule, $P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|B')P(B')} = \frac{0.12}{0.24} = \frac{1}{2}$.

3. Suppose X is a random variable valued in $\{1, 2, 3, 4, 5\}$, all values equally likely. Calculate

   (a) (2 points) $E(X) =$
   Solution: $E(X) = \frac{1+2+3+4+5}{5} = 3$.

   (b) (2 points) $E(X^2) =$
   Solution: $E(X^2) = \frac{1^2+2^2+3^2+4^2+5^2}{5} = 11$.

   (c) (2 points) $\mathrm{Var}(X) =$
   Solution: $\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 2$

   (d) (2 points) $\mathrm{Var}(3X + 2) =$
   Solution:

$$\begin{aligned}
\mathrm{Var}(3X+2) &= E((3X+2)^2) - (E(3X+2))^2 \\
&= E(9X^2 + 12X + 4) - (3E(X) + 2)^2 \\
&= 9E(X^2) + 12E(X) + 4 - 9E(X)^2 - 12E(X) - 4 \\
&= 9\mathrm{Var}(X) \\
&= 18
\end{aligned}$$

4. An urn contains 30 red, 20 black and 50 white balls. Balls are drawn from the urn with replacement. What is the probability that

   (a) (2 points) the second ball is black?
   Solution: Proba (second ball black)=20/100=0.2.

   (b) (2 points) the first black drawn is on the fifth drawn?
   Solution: Proba (first black drawn is on fifth drawn)=$0.8^4.0.2$ (geometric distribution with $p = 0.2$.)

5. A fair die is tossed 720 times independently. Let X be the number of 1's.

   (a) (2 points) What is the probability that $X = 100$? (write the answer in terms of fractions and powers).
   Solution: X has a binomial distribution with parameters $n = 720$ and $= \frac{1}{6}$. Therefore,

$$\mathrm{Proba}(X = 100) = \binom{720}{100} \left(\frac{1}{6}\right)^{100} \left(\frac{5}{6}\right)^{620}.$$

(b) (2 points) Determine $E(X)$ and $\text{Var}(X)$.
Solution: $E(X) = np = \frac{720}{6} = 120$ and $\text{Var(X)} = np(1-p) = \frac{720*5}{36} = 100$.

(c) (3 points) Use the central limit theorem to estimate $P(100 \leq X \leq 130)$. Write the answer in terms of $\Phi(x)$, the cumulative distribution function of the standard normal distribution.
Solution: Denote Z the z-score of X and Y the standard normal random variable. By the Central Limit Theorem, we find

$$\begin{aligned} P(100 \leq X \leq 130) &= P(\frac{100-120}{10} \leq \frac{X-120}{10} \leq 130 - 12010) \\ &= P(-2 \leq Z \leq 1) \\ &\approx P(-2 \leq Y \leq 1) \\ &= \Phi(1) - \Phi(-2). \end{aligned}$$

6. Suppose two random variables X and Y each have possible values 1 and 2 and their joint probability function is $f_{X,Y}(1,1) = 0.4$, $f_{X,Y}(1,2) = 0.2$, $f_{X,Y}(2,1) = 0.3$ and $f_{X,Y}(2,2) = 0.1$.

(a) (2 points) Calculate $\text{Proba}(X+Y=3)$
Solution: $\text{Proba}(X+Y=3) = \text{Proba}(X=2, Y=1) + \text{Proba}(X=1, Y=2) = 0.5$.

(b) (2 points) Determine the marginal probability dstribution of Y.
Solution: We have $f_Y(1) = P(Y=1) = P(X=1, Y=1) + P(X=2, Y=1) = 0.7$ and $f_Y(2) = P(Y=2) = P(X=1, Y=2) + P(X=2, Y=2) = 0.3$

(c) (2 points) Are X and Y independent? Justify your answer.
Solution: $f_{X,Y}(1,1) = 0.4$, $f_Y(1) = 0.7$ and we can easily determine that $f_X(1) = 0.6$. Therefore $f_{X,Y}(1,1) \neq f_Y(1) f_X(1)$ so X and Y are dependent.

7. (2 points) Suppose X is exponentially distributed with parameter $\lambda = 2$. Calculate $P(X > 1)$.
Solution: $P(X>1) = \int_1^{+\infty} 2e^{-2x} dx = -e^{-2x}|_1^{+\infty} = e^{-}2$.

8. (2 points) Let $f(x) = kx$, k a constant, on the interval $[0,3]$.

(a) (2 points) Find the value that makes f a probability density function.
Solution: $k = \frac{1}{\int_0^3 x dx} = \frac{1}{9/2} = \frac{2}{9}$.

(b) (2 points) For the value of k found in the previous question, calculate $E(X)$.
Solution: $E(X) = \frac{2}{9} \int_0^3 x^2 dx = \frac{2}{9} . \frac{x^3}{3} |_0^3 = 2$.

(c) (2 points) For the same value of k, calculate $\text{Proba}(0 \leq X \leq 2)$.
Solution: $\text{Proba}(0 \leq X \leq 2) = \frac{2}{9} \int_0^2 x dx = \frac{4}{9}$.

9. Let X a random discrete variable with the following probability distribution: $P(X=0) = \frac{\theta}{2}$, $P(X=1) = 1-\theta$ and $P(X=2) = \frac{\theta}{2}$, with $0 \leq \theta \leq 1$. The following 4 observations are taken from X: 0, 1, 0, 2.

(a) (2 points) Determine the likelihood function $L(\theta)$?
Solution: $L(\theta) = \frac{\theta^3}{8}(1-\theta)$.

(b) (3 points) Determine the maximum likelihood estimate of θ? (You might want to use the function $G(\theta) = \ln(L(\theta))$. Solution: We want to maximize $L(\theta)$. To do so, we can maximize the function

$$G(\theta) = \ln(L(\theta)) = 3\ln(\theta) + \ln(1-\theta) - \ln(8).$$

Differentiating we find $G'(\theta) = \frac{3}{\theta} - \frac{1}{1-\theta}$ thus $G'(\theta) = 0$ if and only if $\theta = \frac{3}{4}$. Differentiating again, we find $G''(\theta) = -\frac{3}{\theta^2} - \frac{1}{(1-\theta)^2} < 0$ so $\theta = \frac{3}{4}$ is a maximum for $G(\theta)$. Consequently, $\hat{\theta} = \frac{3}{4}$.