Exam 1, Math 472, Spring 2018

Last name, first name, ID number : Score: /30

1. (6 points) Let n be an integer and Y have a Gamma distribution with parameters $\alpha = \frac{n}{2}$ and $\beta > 0$. Show that the random variable $W = \frac{2Y}{\beta}$ has a Chi-square distribution with n degrees of freedom.
   Hint: the random variable W has a Chi-square distribution with n degrees of freedom if and only if its moment generating function is $\psi_W(t) = (1-2t)^{-\frac{n}{2}}$.

   Solution: The moment generating function of W is

$$\begin{aligned}\psi_W(t) &= E[e^{tW}] \\ &= E[e^{\frac{2t}{\beta}W}] \\ &= \psi_W\big(\frac{2t}{\beta}\big) \\ &= \Big(1-\beta\big(\frac{2t}{\beta}\big)\Big)^{-\frac{n}{2}} \\ &= (1-2t)^{-\frac{n}{2}}.\end{aligned}$$

   Thus W has a Chi-square distribution with n degrees of freedom

2. (7 points) Estimate the probability that the number of heads lies between 40 and 60 (both included) when a fair coin is tossed 100 times.
   Hint: If Φ is the c.d.f of $Z \sim \mathcal{N}(0,1)$, then $\Phi(2) = 0.9772$

   Solution: Denote $n = 100$ and S_n the number of heads when the coin is tossed 100 times. The variable S_n is a sum of n Bernouilli Random variables with mean $\mu = \frac{1}{2}$ and standard deviation $\sigma = \frac{1}{2}$. Since n is reasonably large, we can apply the central limit theorem and we get

$$\begin{aligned}P(40 \leq S_n \leq 60) &= P\Big(\frac{40-100.\frac{1}{2}}{\sqrt{10}\frac{1}{2}} \leq \frac{S_n - 100.\frac{1}{2}}{\sqrt{10}\frac{1}{2}} \leq \frac{60-100.\frac{1}{2}}{\sqrt{10}\frac{1}{2}}\Big) \\ &= P\Big(-2 \leq \frac{S_n - 100.\frac{1}{2}}{\sqrt{10}\frac{1}{2}} \leq 2\Big) \\ &\approx P\Big(-2 \leq Z \leq 2\Big) \text{ where } Z \sim \mathcal{N}(0,1) \\ &= 2P\Big(Z \leq 2\Big) \\ &\approx 0.9545\end{aligned}$$

3. Suppose that that $X_1, \ldots, X_n$ form a random sample from a Poisson distribution $P(\lambda)$ whose parameter $\lambda > 0$ is unknown. We assume that λ has an exponential distribution with mean 1 and we observe $X_1 = x_1, \ldots, X_n = x_n$.

   (a) (5 points) What is the posterior distribution of λ?
   Solution:Denote $x = (x_1, \ldots, x_n)$ the vector of observations, $\xi(\lambda)$ the p.d.f. of the prior distribution of λ and $f(x|\lambda)$ the p.f. of the joint distribution of $(X_1, \ldots, X_n)$ conditional to λ. The p.f/p.d.f. of the posterior distribution of λ satisfies

$$\begin{aligned}\xi(\lambda|x) &\propto \xi(\lambda) f(x|\lambda) \\ &\propto e^{-\lambda} e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i} \\ &\propto e^{-(n+1)\lambda} \lambda^{\sum_{i=1}^n x_i}\end{aligned}$$

   This is the non-constant term of the p.d.f. of a Gamma distribution with parameters $\alpha = 1 + \sum_{i=1}^n x_i$ and $\beta = n + 1$, which is the posterior distribution of λ.

(b) (4 points) What is the Bayesian estimate of λ when the square error loss function is used?

Solution: This is the expected value of the posterior distribution, that is $\hat{\lambda} = \frac{1+\sum_{i=1}^n x_i}{n+1}$.

(c) (4 points) Write an integral equation that must be satisfied by the Bayesian estimate of λ when the absolute error loss function is used. (Do not try to solve the equation).

Solution: In that case, the Bayesian estimate of λ is the median of the posterior distribution, which means that $\hat{\lambda}$ satisfies the equation

$$\int_0^{\hat{\lambda}} \frac{(n+1)^{1+\sum_{i=1}^n x_i}}{\Gamma(1+\sum_{i=1} x_i)} e^{-(n+1)\lambda} \lambda^{\sum_{i=1}^n x_i} d\lambda = \frac{1}{2}.$$

4. (7 points) Suppose that $X_1, \ldots, X_n$ form a random sample from the geometric distribution for which the parameter $p \in [0,1]$ is unknown. Suppose also that we observe $X_1 = x_1, \ldots, X_n = x_n$. Find the maximum likelihood estimate of p.

Solution: For all $1 \leq i\ n$, the probability function $P(X_i = x_i)$ is given by

$$P(X_i = x_i) = p(1-p)^{x_i - 1}.$$

Thus the likelihood function is

$$L(p) = f(x|p) = \Pi_{i=1}^n p(1-p)^{x_i-1} = p^n (1-p)^{\sum_{i=1}^n (x_i - 1)}.$$

where $x = (x_1, \ldots, x_n)$ is the vector of observations. The maximum likelihood estimate of p is the solution of the maximization problem

$$\max_p f(x|p)$$

which is also the solution of

$$\max_p \ln(f(x|p))$$

since the natural log function is strictly increasing. Denote $G(p) = \ln(f(x|p))$. Calculating we find

$$G(p) = n \ln(p) + \ln(1-p) \sum_{i=1}^n (x_i - 1)$$

hence

$$G'(p) = \frac{n}{p} - \frac{\sum_{i=1}^n (x_i - 1)}{1-p} = \frac{n}{p(1-p)} - \frac{\sum_{i=1}^n x_i}{1-p}$$

and $G'(p) = 0$ if and only if $p = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{x}_n}$ where $\bar{x}_n$ is the sample mean of the observations. Moreover, we have

$$G''(p) = \frac{n(2p-1)}{p^2(1-p)^2} - \frac{\sum_{i=1}^n x_i}{(1-p)^2} \leq 0$$

since $\sum_{i=1}^n x_i \geq n$. Therefore $G(p)$ has a maximum at $p = \frac{1}{\bar{x}_n}$ which implies that $\hat{p} = \frac{1}{\bar{x}_n}$.