Exam 2, Math 305, Spring 2018

Last name, first name, ID number : Score: /45

1. Suppose X and Y have

$$E(X) = 3,\ E(X^2) = 10,\ E(Y) = 1,\ E(Y^2) = 5,\ E(XY) = 1$$

   (a) (2 points) Find Cov(X,Y)= $E(XY) - E(X)E(Y) = 1 - 3.1 = -2$
   (b) (2 points) Find σ_X= $\sqrt{\text{Var}(X)} = \sqrt{E(X^2) - E(X)^2} = \sqrt{10-9} = 1$
   (c) (2 points) Find σ_Y= $\sqrt{\text{Var}(Y)} = \sqrt{E(Y^2) - E(Y)^2} = \sqrt{5-1} = 2$
   (d) (2 points) Find the dimensionless correlation coefficient $\rho(X,Y)$= $\frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{-2}{1.2} = -1$
   (e) (2 points) Are X and Y independent? Justify your answer.
   Solution: No, since $\text{Cov}(X,Y) \neq 0$.

2. Suppose X and Y have

$$E(X) = 2,\ E(Y) = 3,\ E(XY) = 6$$

   (a) (2 points) Find Cov(X,Y)= $E(XY) - E(X)E(Y) = 6 - 3.2 = 0$
   (b) (2 points) Are X and Y independent? Justify your answer.
   Solution:Not necessarily, since $\text{Cov}(X,Y) = 0$ does not necessarily imply that X and Y are independent.
   (c) (2 points) Set $Z = 2X - 1$. Find the dimensionless correlation coefficient $\rho(X,Z)$.
   Solution: Since $Z = aX + b$ where $a = 2 > 0$, then $\rho(X,Z) = 1$.
   (d) (2 points) Set $T = -Y+1$. Find the dimensionless correlation coefficient $\rho(Y,T)$. Solution: Since $T = aY+b$ where $a = -1 < 0$, then $\rho(Y,T) = -1$.

3. In an oral exam, you are asked a sequence of independent questions of equal difficulty. You start with 5 points. You earn 1 point for each question you get right and lose 1 point for each question you get wrong. If you reach 0, the exam is over and you get a F. If you reach 10, the exam is over and you get an A. You are allowed infinite time for this exam.

   (a) (6 points) If, for each question, your probability of answering correctly is 0.6, what is the probability that you will get an A?
   Solution This is a simple ramdom walk process. We want to calculate the probability, denoted u_5, defined by

$$u_5 = P(\text{ We have 5 points and we reach 10 before 0})$$

   if

$$p = \text{probability of answering a question correctly} = 0.6$$

   and

$$q = \text{probability of answering a question wrong} = 1 - p = 0.4.$$

   Since $p \neq \frac{1}{2}$, we know that

$$u_5 = \frac{1 - (\frac{0.4}{0.6})^5}{1 - (\frac{0.4}{0.6})^{10}} = \frac{1 - (\frac{2}{3})^5}{1 - (\frac{2}{3})^{10}}$$

   (b) (4 points) If, for each question, your probability of answering correctly is 0.5, what is the probability that you will get an A?
   Solution: This is exactly the same question as question (a) with $p = 1 - p = \frac{1}{2}$. In that case, we have $u_5 = \frac{5}{10} = \frac{1}{2}$

4. (6 points) Assume $N(t)$ is the population at time t. Assume that there are two types of events, births and deaths, which occur independently of each other, and

$$P(1 \text{ birth in } (t, t+h) \mid N(t) = n) = n\beta h + o(h)$$
$$P(1 \text{ death in } (t, t+h) \mid N(t) = n) = n\delta h + o(h)$$
$$P(\text{more than 1 event in } (t, t+h) \mid N(t) = n) = o(h)$$
$$P(0 \text{ event in } (t, t+h) \mid N(t) = n) = 1 - n(\beta + \delta)h + o(h)$$

Find the Kolmogorov differential equation for $p_n(t) = P(N(t) = n)$.
Solution: We have

$$\begin{aligned} p_n(t+h) &= P(N(t+h) = n) \\ &= P\big(N(t) = n \text{ and } 0 \text{ event in } (t, t+h)\big) + P\big(N(t) = n-1 \text{ and } 1 \text{ birth in } (t, t+h)\big) \\ &\quad + P\big(N(t) = n+1 \text{ and } 1 \text{ death in } (t, t+h)\big) \\ &= \big(1 - n(\beta+\delta)h\big)p_n(t) + (n-1)\beta h p_{n-1}(t) + (n+1)\delta h p_{n+1}(t) + o(h) \end{aligned}$$

thus

$$p_n(t+h) - p_n(t) = -n(\beta+\delta)hp_n(t) + (n-1)\beta h p_{n-1}(t) + (n+1)\delta h p_{n+1}(t) + o(h)$$

which leads to

$$\frac{p_n(t+h) - p_n(t)}{h} = -n(\beta+\delta)p_n(t) + (n-1)\beta p_{n-1}(t) + (n+1)\delta p_{n+1}(t) + \frac{o(h)}{h}$$

hence and

$$p_n'(t) = -n(\beta+\delta)p_n(t) + (n-1)\beta p_{n-1}(t) + (n+1)\delta p_{n+1}(t)$$

which is the Kolmogorov equation for $p_n(t)$.

5. Suppose random variables $X_k,\ k = 0,\ 1, 2, \ldots$ are valued in $\{0, 1, \ldots, N\}$. Consider the simple random walk with $p = P(X_{k+1} = n+1 | X_k = n)$ and $1 - p = P(X_{k+1} = n-1 | X_k = n)$. Denote T_n the time it takes to reach a boundary assuming $X_0 = n$. Let $v_n = E(T_n)$.

   (a) (4 points) Find a difference equation for v_n.
   Solution: Denote R the event "moving to the right" and L the event "moving to the left". Then p=P(R) and 1-p=P(L). Thus

$$\begin{aligned} v_n &= E(T_n|R)p + E(T_n|L)(1-p) \\ &= (v_{n+1} + 1)p + (v_{n-1} + 1)(1-p) \\ &= v_{n+1}p + v_{n-1}(1-p) + 1. \end{aligned}$$

   (b) (2 points) Find the boundary conditions.
   Solution: We have

$$\begin{aligned} v_0 &= E(T_0) = 0 \\ v_N &= E(T_N) = 0. \end{aligned}$$

6. (5 points) Find the solution of the difference equation

$$2u_{n+2} - 3u_{n+1} + u_n = 0$$

subject to the boundary conditions u_0 and $u_4 = 15$.
Solution: Let $u_n = \lambda^n$, with $\lambda \neq 0$. Plugging into the difference equation and dividing by λ^n, we get

$$2\lambda^2 - 3\lambda + 1 = 0$$

whose solution are $\lambda = \frac{1}{2},\ 1$. Thus $u_n = 1$ and $u_n = \left(\frac{1}{2}\right)^n$ are two particular solutions so the general solution is $u_n = a + b\left(\frac{1}{2}\right)^n$ where a and b are two arbitrary constants. We want $u_0 = a + b = 0$ so $a = -b$ so $u_n = a\left(1 - \left(\frac{1}{2}\right)^n\right)$. Moreover, we want $u_4 = a\left(1 - \frac{1}{16}\right) = 15$ so $a = 16$. Therefore, the unique solution of the equation is

$$u_n = 16\left(1 - \left(\frac{1}{2}\right)^n\right)$$

7. (4 points) Find the solution of the differential equation $y'(t) = -2y(t) + 2$ with $y(0) = 0$.
<u>Solution</u>: The Integrating factor is e^{2t}. Multiplying both sides by the Integrting factor, using the produt rule, integrating and solving for $y(t)$, we find the general solution

$$y(t) = 1 + Ce^{-2t}$$

where C is an arbitrary constant. For $y(0) = 1$, we need $c = -1$. Thus, the unique solution is

$$y(t) = 1 - e^{-2t}.$$