Exam 2, Math 472, Spring 2018

Last name, first name, ID number : Score: /30

1. (2 points) Suppose that $X_1, \dots, X_n$ form a random sample from a normal distribution $\mathcal{N}(\mu, \sigma^2)$ where μ is known and σ^2 is unknown. Let $U = \sum_{i=1}^{n}(X_i - \mu)^2$.

   (a) (2 points) Find the distribution of $W = \frac{U}{\sigma^2}$
   Solution: We have $W = \sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma}\right)^2$. For all $1 \le i \le n$, $\frac{X_i-\mu}{\sigma} \sim \mathcal{N}(0,1)$ (corollary 5.6.1), thus $\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(1)$ (thm 8.2.3) and $W \sim \chi^2(n)$ (thm 8.2.2).

   (b) (2 points) Find $E[U]$.
   Solution: We have $E[U] = E[\sigma^2 W] = \sigma^2 E[W] = n\sigma^2$.

   (c) (3 points) Use U to construct an unbiased estimator for σ^2.
   Solution: We have $E[\frac{U}{n}] = \frac{E[U]}{n} = \sigma^2$. Thus $\frac{U}{n} = \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2$ is an unbiased estimator for σ^2.

2. Let $Y_1, \dots, Y_n$ form a random sample from a normal distribution $\mathcal{N}(\mu, \sigma^2)$. Assume that $n = 2k$, for some integer k, and define the estimator $\hat{\sigma^2}$ for, σ^2, by

$$\hat{\sigma^2} = \frac{1}{2k}\sum_{i=1}^{k}(Y_{2i} - Y_{2i-1})^2.$$

   (a) (4 points) Show that, for all $1 \le i \le k$, $\frac{(Y_{2i}-Y_{2i-1})^2}{2\sigma^2} \sim \chi^2-$distribution with 1 degree of freedom.
   Solution: We have $\frac{Y_{2i}-Y_{2i-1}}{\sqrt{2}\sigma} \sim \mathcal{N}(0,1)$ (corollary 5.6.1)so $\frac{(Y_{2i}-Y_{2i-1})^2}{2\sigma^2} \sim \chi^2-$distribution with 1 degree of freedom (theorem 8.2.3).

   (b) (4 points) Use the result from the question (a) to show that, for all $1 \le i \le k$, $(Y_{2i} - Y_{2i-1})^2 \sim$ Gamma-distribution with parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4\sigma^2}$. (Hint: use moment generating functions).
   Solution: According to the result of question (a), we have $(Y_{2i} - Y_{2i-1})^2 = 2\sigma^2 X$ where $X \sim \chi^2-$distribution with 1 degree of freedom. Denote $\psi(t)$ the m.g.f of $(Y_{2i} - Y_{2i-1})^2$ and $\psi_X(t)$ the m.g.f. of X. We have

$$\begin{aligned}\psi(t) =& E(e^{t2\sigma^2 X}) \\ &= \psi_X(2\sigma^2 t) \\ &= \left(\frac{1}{1-4\sigma^2 t}\right)^{\frac{1}{2}} \\ &= \left(\frac{1/4\sigma^2}{1/4\sigma^2 - t}\right)^{\frac{1}{2}}\end{aligned}$$

   which is the m.g.f. of a Gamma-distribution with parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4\sigma^2}$.

   (c) (4 points) Show that $\hat{\sigma^2}$ is an unbiased estimator for σ^2
   Solution: we have

$$E(\hat{\sigma^2}) = \frac{1}{2k}\sum_{i=1}^{k} E((Y_{2i} - Y_{2i-1})^2) = \frac{1}{2k}\sum_{i=1}^{k} 2\sigma^2 = \sigma^2.$$

   Hence $\hat{\sigma^2}$ is an unbiased estimator for σ^2. %item (4 points) Show that $\hat{\sigma^2}$ is a consistent estimator for σ^2. (Hint: since $Y_1, \dots, Y_n$ form a random sample, the random variables $Y_{2i} - Y_{2i-1}$ are mutually independent.)
   Solution: Since the variables $Y_{2i} - Y_{2i-1}$ are mutually independent, we have

$$\mathrm{Var}(\hat{\sigma^2}) = \frac{1}{4k^2}\sum_{i=1}^{k} \mathrm{Var}((Y_{2i} - Y_{2i-1})^2) = \frac{2\sigma^4}{k} = \frac{4\sigma^4}{n} \xrightarrow{n\to\infty} 0.$$

Hence, according to the theorem of consistency of unbiased estimators, $\hat{\sigma^2}$ is a consistent estimator for σ^2

3. (5 points) Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution $\mathcal{N}(\mu, \sigma^2)$. Denote $\hat{\sigma^2}$ the sample variance. Use one of the probability tables at the end of the book to determine the smallest value of n for which $P(\frac{\hat{\sigma^2}}{\sigma^2} \leq 1.5) \geq 0.95$.
<u>Solution</u>: We have

$$P(\frac{\hat{\sigma^2}}{\sigma^2} \leq 1.5) = P(\frac{n\hat{\sigma^2}}{\sigma^2} \leq 1.5n) = P(V \leq 1.5n)$$

where $V = \frac{n\hat{\sigma^2}}{\sigma^2} \sim \chi^2(n-1)$ (thm 8.3.1). Let's use the probability table of the χ^2-distribution (p 858/859 of the textbook). When $n = 20$, then $V \sim \chi^2(19)$, $1.5n = 30$ and we read from the table that $P(V \leq 30) \leq P(V \leq 30.14) = 0.95$ so the condition is not met. When $n = 21$, then $V \sim \chi^2(20)$, $1.5n = 31.5$ and we read from the table that $P(V \leq 31.5) \geq P(V \leq 31.41) = 0.95$ so the condition is met. Thus, the smallest value of n is 21.

4. (5 points) Suppose that $X_1, \ldots, X_{16}$ form a random sample from a normal distribution $\mathcal{N}(\mu, \sigma^2)$. Denote $\bar{X}_{16}$ the sample mean and σ' the unbiased sample variance. Let c be the real number such that $P(\bar{X_{16}} - \frac{c\sigma'}{4} < \mu < \bar{X_{16}} + \frac{c\sigma'}{4}) = 0.95$. Express c in terms of 0.95 and the cumultative distribution function of a $t-$distribution.
<u>Solution</u>: Notice that

$$P\big(\bar{X_{16}} - \frac{c\sigma'}{4} < \mu < \bar{X_{16}} + \frac{c\sigma'}{4}\big) = 0.95 \iff P\big(-c < \frac{4(\bar{X_{16}} - \mu)}{\sigma'} < c\big) = 0.95.$$

According to theorem 8.4.2, the random variable $\frac{4(\bar{X_{16}}-\mu)}{\sigma'}$ has a t-distribution with 15 degrees of freedom. Denote T_{15} the c.d.f. of the t-distribution with 15 degrees of freedom. By symmetry of the t-distributions, we have

$$P\big(\bar{X}_n - \frac{c\sigma'}{4} < \mu < \bar{X}_n + \frac{c\sigma'}{4}\big) = 0.95 \iff T_{15}(c) - T_{15}(-c) = 0.95 \iff c = T_{15^{-1}}\Big(\frac{1.95}{2}\Big).$$