EVERY INFINITE ORDER MAPPING CLASS HAS AN INFINITE ORDER ACTION ON THE HOMOLOGY OF SOME FINITE COVER

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ABSTRACT. We prove the following well known conjecture: let $\Sigma$ be an oriented surface of finite type whose fundamental group is a nonabelian free group. Let $\phi \in \text{Mod}(\Sigma)$ be a an infinite order mapping class. Then there exists a finite solvable cover $\Sigma_0 \to \Sigma$, and a lift $\phi_0$ of $\phi$ such that the action of $\phi_0$ on $H_1(\Sigma_0, \mathbb{Z})$ has infinite order. Our main tools are the theory of homological shadows, which was previously developed by the author, and Fourier transforms.

1. Introduction

Let $\Sigma$ be an oriented surface of finite type, and let $\text{Mod}(\Sigma)$ be its mapping class group. The finite dimensional representation theory of $\text{Mod}(\Sigma)$ has been the subject of much study, and much about it remains mysterious. For instance, it is not generally known whether the groups $\text{Mod}(\Sigma)$ are linear. On the other hand, these groups have extensive collections of finite dimensional representations.

The largest and most tractable collection of representations are the homological representations which are associated to finite covers of $\Sigma$. The first of these is the standard homological representation $\text{Mod}(\Sigma) \to \text{GL}(H_1(\Sigma, \mathbb{Z}))$ given by the action on first homology. The kernel of this representation is called the Torelli group.

More generally, pick a base point $\beta \in \Sigma$ and let $\Sigma'$ be the surface $\Sigma$ punctured at $\beta$. If $\pi : \Sigma_0 \to \Sigma$ is a finite cover, there is a finite index subgroup $M_0 \leq \text{Mod}(\Sigma', \beta)$ of mapping classes fixing $\beta$ that lift to $\Sigma_0$. This gives a representation $M_0 \to \text{Mod}(\Sigma_0) \to \text{GL}(H_1(\Sigma_0, \mathbb{Z}))$, that can be induced to a representation $\rho_\pi$ of $\text{Mod}(\Sigma')$. Thus, for each finite cover of $\Sigma$, we construct a finite dimensional representation of $\text{Mod}(\Sigma')$.

Homological representations are a very rich family of representations. For example, in [9] Grunewald, Larsen, Lubotzky and Malestein show that the mapping class group of a once punctured surface has several infinite families of

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arithmetic groups as images of homological representation (In [8], Grunewald and Lubotzky show a similar result for homological representations of Aut($F_n$), which are constructed in a similar way). Looijenga proves a similar result for abelian covers in [14].

These representations contain a great deal of information about the mapping class group. For instance in [21] Putman and Wieland exhibit a beautiful connection between properties of homological representations and the virtual first Betti number of mapping class group. In [18] McMullen studies the structure of genus 0 homological representations extensively and explores their strong connections to the study of moduli spaces of Riemann surfaces.

Furthermore, due to the fact that homological representations are the only class of representations of Mod($\Sigma$) whose images are well understood, they often come up in applications where representations of mapping class group with large images are needed. One such application is the study of properties of random elements of subgroups of Mod($\Sigma$) (see for example [15], [16], [17], [22]).

Aside from containing information about the group as a whole, these representations contain information about individual elements. It is a simple exercise to show that for any nontrivial $\phi \in$ Mod($\Sigma$) there is a cover $\pi$ in which $\rho_\pi(\phi)$ is non trivial. Koberda and later Koberda and Mangahas showed that homological representations can detect the Nielsen-Thurston classification of a mapping class ([12], [13]).

Despite being a well studied family, homological representations still remain quite mysterious and many surprisingly basic and natural questions about them remain unanswered. One such question is the following.

**Question 1.1.** Suppose $\phi \in$ Mod($\Sigma$) has infinite order. Is it possible to find a cover $\pi$ such that $\rho_\pi(\phi)$ has infinite order?

McMullen considered a stronger version of question 1.1 Let $\phi$ be a pseudo-Anosov mapping class with dilatation $\lambda$. For any finite cover $\pi$, let $\sigma_\pi$ be the spectral radius of $\rho_\pi(\phi)$. It is easy to show that $\log(\sigma_\pi) \leq \lambda$. For pseudo-Anosovs with orientable foliations, it’s true that $\lambda = \log \sigma_I$, where $I$ is the trivial cover. For some time it was conjectured that for any pseudo-Anosov map $\lambda = \sup \log \sigma_\pi$, where the supremum is taken over all finite covers. In [19], McMullen showed that this is not the case for a large class of pseudo-Anosov maps. McMullen then asked the following question.

**Question 1.2.** In the notation above, is $\sup \log \sigma_\pi > 0$? In other words, can one always find a finite cover where the action on homology has an eigenvalue off of the unit circle?

In this paper we provide a positive answer to question 1.1 for mapping classes of punctured surfaces.
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Theorem 1.3. Let \( \Sigma \) be an oriented surface of finite type whose fundamental group is free. Let \( \phi \in \text{Mod}(\Sigma) \) be an infinite order mapping class. Then there exists a finite characteristic solvable cover \( \Sigma_0 \to \Sigma \), and a lift \( \phi_0 \) of \( \phi \) such that the action of \( \phi_0 \) on \( H_1(\Sigma_0, \mathbb{Z}) \) has infinite order.

We also prove a related theorem for automorphisms of free groups.

Theorem 1.4. Let \( \phi \in \text{Out}(F_n) \) be a fully irreducible automorphism. Then there exists a representative \( f \) of \( \phi \) in \( \text{Aut}(F_n) \) and an \( f \)-invariant finite index characteristic subgroup \( K < F_n \) such that the action of \( f|_K \) on \( H_1(K, \mathbb{Z}) \) has infinite order, and \( F_n/K \) is solvable.

The restriction that \( \Sigma \) have a free fundamental group is rather technical. We conjecture that our proof can be strengthened to cover the case of closed surfaces.

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1.1. Sketch of proof. We sketch the proof in the case where \( f \in \text{Aut}(\pi_1(\Sigma)) \) is induced by a pseudo-Anosov diffeomorphism, and acts trivially on \( H_1(\Sigma, \mathbb{Z}) \). Part of the difficulty in the proof arises from the fact that we are searching for a finite cover with certain properties, and that the set of all such finite covers is extremely large. To circumvent this problem, we will always try to look for abelian covers, which can be parametrized as rational points on a torus. We will end up taking a sequence of such covers.

Step 1 - passing to abelian covers. We begin by considering an infinite abelian cover that contains information about all finite abelian covers. Let \( R = \bigvee_1^n S^1 \) be a wedge of \( n \) circles. We have that \( \pi_1(R) = F_n \). Pick a continuous map \( \varphi : R \to R \) inducing the automorphism \( f \). Let \( \tilde{R} \to R \) be the universal abelian cover that is the cover corresponding to \([F_n, F_n]\). The space \( \tilde{R} \) is an \( n \)-dimensional grid.

Let \( G = C_1(\tilde{R}, \mathbb{Z}) \) be the space of 1-chains in \( \tilde{R} \). This is the space of formal combinations of edges in \( \tilde{R} \) with coefficient in \( \mathbb{Z} \). The reason we work with 1-chains as opposed to homology classes is that the space \( G \) has a particularly nice algebraic description - if we set \( H = H_1(F_n, \mathbb{Z}) \), then \( G \cong \mathbb{Z}[H]^n \).

The map \( \varphi \) can be lifted to a map \( \tilde{\varphi} \) that acts on \( G \). This action also has a particularly nice algebraic description - it is given by multiplication by a matrix \( A_{\varphi} \in M_n(\mathbb{Z}[H]) \). This matrix is the image of \( f \) under the Magnus representation, which was originally defined by Satoh (see [23] for a survey).

The matrix \( A_{\varphi} \) contains information about the action of \( f \) on finite abelian covers. Suppose \( \psi : H \to \mathbb{C}^n \) is a homomorphism with finite image. We can
think of \( \psi \) as a homomorphism from \( F_n \to \mathbb{C}^n \). Let \( r = r_{\psi} = \#\psi(F_n) \) be the size of the image of \( \psi \). Denote by \( R_{\psi} \) the cover corresponding to the kernel of the natural map \( F_n \to H_1(F_n, \mathbb{Z}/n\mathbb{Z}) \). Note that this is always a characteristic cover.

The map \( \psi \) can be thought of as an assignment of complex values to a set of generators of \( H \). Since the coordinates of \( A_{\varphi} \) are polynomials in these generators, we can evaluate \( A_{\varphi} \) at \( \psi \) by plugging in the values into these polynomials. In this way we get a matrix \( \psi(A_{\varphi}) \in M_n(\mathbb{C}) \). This matrix gives the action of a lift of \( \varphi \) on a subspace of \( C_1(R_{\psi}, \mathbb{Z}) \), which can be identified with the twisted chain space \( C_1(R, \mathbb{C}_{\psi}) \).

We show that if this action has eigenvalues off of the unit circle, then so does the action of a lift of some power of \( f \) on \( H_1(R_{\psi}, \mathbb{Z}) \). Thus, it is enough to show that for some \( \psi \), the matrix \( \psi(A_{\varphi}) \) has eigenvalues off of the unit circle. One way to do this is to find a \( \psi \) such \( |\psi(\text{Tr}(A_{\varphi}))| > n \). This is the content of Proposition 2.5. We will find such a \( \psi \) by studying the geometry of the support of \( \text{Tr}(A_{\varphi}) \).

**Step 2 - homological shadows.** We study the geometry of the support of \( \text{Tr}(A_{\varphi}) \) using the technology of homological shadows. Given a word \( w \in F_n \), the homological shadow of \( w \) or \( S_w \) is the set of vertices of \( \tilde{R} \) through which a lift of \( w \) passes. Since the vertices of \( \tilde{R} \) correspond to the elements of \( H_1(F_n, \mathbb{Z}) \), we can think of \( S_w \) as a subset of \( H \otimes \mathbb{R} \cong \mathbb{R}^n \). In [10] we showed that there exists an \( n \)-dimensional convex polytope \( Sf \in \mathbb{R}^n \) such that for any word \( w \) with infinite \( f \) orbit, \( \lim_{k \to \infty} \frac{1}{k} Sf^k(w) = Sf \). This technology is discussed in Section 2.3.

**Step 3 - stable covers.** Let \( v \) be a vertex of \( Sf \). Suppose \( R_v \to R \) is a finite cover to which \( f \) lifts, such that the action of \( f \) on \( H_1(R_v, \mathbb{Z}) \) is trivial and and a term projecting to \( v \) appears in \( \text{Tr}(A_{\varphi_v}) \), where \( \varphi_v \) denotes a lift of \( \varphi \) to the cover \( R_v \), (we will actually use a slightly more technical definition for stable vertices which will be better suited for our needs). In this case we say that \( v \) is stable in \( R_v \). If all the vertices are stable, then the cover is called stable. In Proposition 2.18 we use Fourier transform techniques to show that if we find a stable cover to which \( f \) lifts, then we can find a \( \psi \) that satisfies the condition of Proposition 2.5, and thus find a cover where the homological action of \( f \) has infinite order.

**Step 4 - stabilizing vertices.** The remainder of the proof reduces to the following problem - given a vertex \( v \) of \( Sf \), find a cover in which it is stable. We do this in two stages, which are covered in Section 2.5.
To every vertex \( v \) we assign an integer measuring its complexity. Roughly speaking, this integer detects how deep into the lower central series of \( F_n \) we need to go before being able to detect the vertex \( v \). A vertex is stable if we assign it the integer 1. The assignment of this integer is the content of Lemma 2.30.

We then show that for every vertex \( v \), there is a finite abelian cover that either reduces its complexity or has an infinite order homological action by a lift of \( f \). This is done in Lemma 2.31. Repeated application of this process allows us to find a cover where the vertex is stable.

2. Proof

The main difficulty in proving Theorem 1.3 is to prove it for pseudo-Anosov mapping classes. To accomplish this, we will prove a slightly more general form of Theorem 1.4. After some initial reductions, the proof of Theorem 1.3 will be almost the same as this case.

Theorem 2.1. Let \( \phi \in \text{Out}(F_n) \) be an outer automorphism that has a train track representative with Perron-Frobenius transition matrix. Then there exists a representative \( f \) of \( \phi \) in \( \text{Aut}(F_n) \) and a characteristic finite index subgroup \( K \triangleleft F_n \) such that the action of \( f \) on \( H_1(K, \mathbb{Z}) \) has infinite order and \( F_n/K \) is solvable.

Note that a pseudo-Anosov diffeomorphism \( \varphi \in \text{Mod}(S) \) induces a well defined element \( \phi_* \in \text{Out}(\pi_1(S)) \), which has a train track representative with Perron-Frobenius transition matrix (see [2] for definitions). We now proceed to prove Theorem 2.1.

Remark 2.2. Throughout the proof, we will often find it necessary to replace \( f \) with a power of itself. Since the finite covers we will be considering are always characteristic, both \( f \) and any power of \( f \) lift to these covers. If the action on homology of some power of \( f \) has infinite order, then so does the action of \( f \). Thus, it is enough to prove Theorem 2.1 for some power of \( f \).

Remark 2.3. If the action of \( f \) on \( H_1(F_n, \mathbb{Z}) \) has infinite order, then Theorem 2.1 is trivial. If the action of \( f \) on \( H_1(F_n, \mathbb{Z}) \) has finite order, we can replace \( f \) with a power of itself that acts trivially. Thus, we will always assume that \( f \) is in the Torelli group, so \( f \) acts trivially on \( H_1(F_n, \mathbb{Z}) \).

2.1. Notation and preliminaries. Pick an automorphism \( f \) that projects to \( \phi \). Let \( (\Gamma, \beta) \) be a graph with fundamental group \( F_n \) (which we will specify later on in the proof) together with a chosen base point \( \beta \in V(\Gamma) \). Denote \( m = \#E(\Gamma) \). Fix, once and for all, an orientation on each of the edges of \( \Gamma \).
Let $\varphi : (\Gamma, \beta) \to (\Gamma, \beta)$ be a continuous function that induces the automorphism $f$. This map can be taken to send any edge to a concatenation of edges.

Let $\tilde{\Gamma} \to \Gamma$ be the universal abelian cover of $\Gamma$, that is, the cover associated to the subgroup $[F_n, F_n]$. Denote $H = H_1(\Gamma)$ (which we will think of as a multiplicative group), and $G = C_1(\tilde{\Gamma}, \mathbb{Z})$, the group of 1-chains in $\tilde{\Gamma}$. The action of $H$ on $\tilde{\Gamma}$ by deck transformations gives an action of $H$ on $G$, which turns it into a $\mathbb{Z}[H]$ module.

In fact, we have the $\mathbb{Z}[H]$-module isomorphism $G \cong \mathbb{Z}[H]^{E(\Gamma)}$. We construct such an isomorphism explicitly. Pick once and for all an orientation on each edge of $\Gamma$. Label the edges of $\Gamma$ $e_1, \ldots, e_m$. Every edge in the graph $\tilde{\Gamma}$ projects to some edge $e_i$ of $\Gamma$. Our choice of a base point $\tilde{\beta}$ of $\tilde{\Gamma}$ gives a correspondence between the lifts of the edge $e_i$ and the elements of $H$. Thus, to each such lift we can uniquely assign an element of $H$. Given such an edge $\eta$, we denote the element of $h$ assigned to it by $h(\eta)$.

Any chain $\eta \in G$ can be written as a formal sum of edges of $\tilde{\Gamma}$:

$$\eta = \sum_{i=1}^{m} \sum_{j} a_{ij} \eta_{ij}$$

where each edge $\eta_{ij}$ is a lift of $e_i$. The function:

$$\bar{h}(\eta) = \left( \begin{array}{c} \sum_j a_{1j} h(\eta_{1j}) \\ \vdots \\ \sum_j a_{mj} h(\eta_{mj}) \end{array} \right) \in (\mathbb{Z}[H])^m$$

gives the isomorphism described above.

2.1.1. The lifted action of $\varphi$ on $G$. The automorphism $f$ acts on $H$. This extends to a ring endomorphism of $\mathbb{Z}[H]$ which we always denote by $f_*$. Throughout the proof we assume that $f$ is in the Torelli subgroup, so $f_* = I_{n}$. Pick a lift $\tilde{\beta}$ of $\beta$, and lift $\varphi$ to a continuous map $\tilde{\varphi} : (\tilde{\Gamma}, \tilde{\beta}) \to (\tilde{\Gamma}, \tilde{\beta})$. The map $\tilde{\varphi}$ induces a homomorphism of $G \cong \mathbb{Z}[H]^{E(\Gamma)}$ given by

$$\tilde{\varphi}_* \left( \begin{array}{c} h_1 \\ \vdots \\ h_m \end{array} \right) = A\varphi \left( \begin{array}{c} h_1 \\ \vdots \\ h_m \end{array} \right)$$

for some matrix $A\varphi \in M_m(\mathbb{Z}[H])$. The assignment $f \to A\varphi$ is called the Magnus representation of the Torelli group.

We need an explicit description of the he matrix $A\varphi$. This is usually done using Fox calculus (see [24] for details). Since our proof uses several variants
of this matrix, we choose to describe the construction in a somewhat indirect way via a directed graph (or digraph) called the *transition graph of* $\varphi$.

2.1.2. The transition graph of $\varphi$ and the matrix $A_\varphi$. Let $\mathcal{T} = \mathcal{T}_\varphi$ be the following directed graph. Set $V(\mathcal{T}) = E(\Gamma)$. Let $e \in V(\mathcal{T})$. Write $\varphi(e) = e_1 \ldots e_s$, where each $e_i$ is an edge of $\Gamma$. Add $s$ edges to $\mathcal{T}$ emanating from $e$, where the $i^{th}$ edge, which we denote $\eta_i$, connects $e$ to the vertex $e_i$.

Let $\overline{h} : G \to \mathbb{Z}[H]^{E(\Gamma)}$ be the isomorphism specified above. For any edge $e$ let $x_e$ be the pre image under $\overline{h}$ of the element with 1 in the $e$-coordinate, and 0 in all others. Let $\tilde{\gamma}_1 \ldots \tilde{\gamma}_s$ be the path $\tilde{\varphi}x_e$. Write this path as $\tilde{w}\tilde{g}_i \pm \tilde{u}$, where $\tilde{g}_i$ is the edge corresponding to $\tilde{\gamma}_i$, traversed in the positive direction.

We associate the following objects to the edge $\eta_i$.

(a) The *sign* of $\eta_i$, or $s(\eta_i) \in \{1, -1\}$, defined to be 1 if $\tilde{g}_i = \tilde{e}_i$ and $-1$ otherwise. That is, the sign of $\eta_i$ denotes whether or not the corresponding edge of $\varphi(e)$ is traversed in the positive or negative direction.

(b) All but one of the coordinates of the vector $\overline{t}(\tilde{g}_i)$ are 0. We denote the remaining coordinate $t(\eta_i)$, and call it the *translation* of $\eta_i$. The translation $t(\eta_i)$ is the element in $H$ corresponding to the beginning (or end, as determined by the sign) of the lift of the $i^{th}$ edge of $\varphi(e)$.

To a cycle $\gamma = \eta_1 \ldots \eta_k$ in $\mathcal{T}$ we associate the following objects.

(a) The *sign* of $\gamma$, or $s(\gamma) = \prod_{i=1}^{k} s(\eta_i)$.

(b) The *translation* of $\gamma$ or $t(\gamma) = \prod_{i=1}^{k} t(\eta_i)$ and the *normalized translation* of $\gamma$ or $t_n(\gamma) = \frac{1}{k}t(\gamma)$, where the latter is thought of as an element of $H \otimes \mathbb{R}$. In the cases we will be considering, we get the simpler expression $t(\gamma) = \prod_{i=1}^{k} t(\eta_i)$.

Note that throughout this paper, we employ the word cycle in a somewhat nonstandard way to mean a sequence of edges originating and terminating at the same point, *not up to cyclical re-ordering of the edges*. Our cycles will always be based cycles. While this does not affect the definitions given above, it will come up in the sequel.

We are now able to give a description of $A_\varphi$, that follows directly from the definitions. Given $e,e' \in E(\Gamma)$, we have that $(A_\varphi)_{e,e'} = \sum s(\eta)t(\eta)$, where the sum is taken over all edges $\eta$ of $\mathcal{T}$ connecting $e$ to $e'$. Notice that in this sum, we are viewing $H$ as a subset of $\mathbb{Z}[H]$ in the obvious way.

2.1.3. Matrices associated to subgraphs of $\mathcal{T}$ and to covers. We will require two variants of the matrix $A_\varphi$. The first variant is obtained by considering subgraphs of $\mathcal{T}$. Let $\mathcal{T}'$ be a subgraph of $\mathcal{T}$. We construct a matrix $A_\varphi[\mathcal{T}']$ by setting:
\((A_\varphi[T'])[e,e'] = \sum_{\eta=(e,e') \in E(T')} s(\eta)t(\eta)\)

The second variant involves covers. Let \(\pi_0 : (\Gamma_0, \beta_0) \to (\Gamma, \beta)\) be a finite cover to which \(\varphi\) can be lifted, and let \(\varphi_0\) be a lift of \(\varphi\) such that \(\varphi_0(\beta_0) = \beta_0\). Let \(T_0\) be the directed graph assigned to \(\varphi_0\). Then the graph \(T_0\) is a finite cover of \(T\), where the covering map respects the directionality of edges. If \(T'\) is a subgraph of \(T\), let \(T'_0\) be its lift to \(T_0\). We will use the notation \(A_\varphi[T', \pi_0]\) for the matrix associated to \(T'_0\).

2.2. Abelian covers and \(\text{Tr}(A_\varphi)\).

2.2.1. The polynomial \(\Delta\) and its specializations. Let:

\[\Delta = \Delta_\varphi(t) = \det(tI - A_\varphi) \in \mathbb{Z}[H][[t]]\]

be the characteristic polynomial of \(A_\varphi\). Given a homomorphism \(\psi : H \to S^1\), we define the specialization of \(\Delta\) at \(\psi\), or \(\Delta_\psi\) by taking the image of \(\Delta\) under the map \(\mathbb{Z}[H][[t]] \to \mathbb{Z}[\psi(H)][[t]]\). Similarly, we define \((A_\varphi)_\psi \in M_m(\mathbb{C})\) to be the image of \(A_\varphi\) under this map. We use the letter \(\Delta\) in analogy with the Alexander polynomial (which is often also denoted \(\Delta\)), whose specializations are useful for the studying the action of lifts of a map on the homology of abelian covers (See [20] for details.)

Suppose that \(\psi : H \to S^1 \subset \mathbb{C}^*\) has finite image. We abuse notation and view \(\psi\) as a map from \(F_n\) to \(\mathbb{C}\). Let \(r_\psi = \#\psi(F_n)\) and let \(\Gamma_\psi\) be the cover corresponding to the kernel of the natural map \(F_n \to H_1(F_n, \mathbb{Z}/n\mathbb{Z})\). Note that this is a characteristic cover.

The group \(F_n\) acts on \(C_1(\Gamma_\psi, \mathbb{C})\) via its action on \(\Gamma_\psi\) by deck transformations. This factors through an action of \(H\) on \(C_1(\Gamma_\psi, \mathbb{C})\). Consider the subspace \(C_\psi \subset C_1(\Gamma_\psi, \mathbb{C})\) defined by

\[C_\psi = \{a \in C_1(\Gamma_\psi, \mathbb{C}) \mid a(\pi) = \psi((a)\pi, \forall a \in F_n)\}\]

The space \(C_\psi\) is an \(m\)-dimensional subspace of \(C_1(\Gamma_\psi, \mathbb{C})\). Indeed, suppose \(e_1, \ldots, e_m\) are the edges of \(\Gamma\). For each \(i\), choose a lift \(\tilde{e}_i\) of \(e\) in \(\Gamma_\psi\). Then the space \(C_\psi\) is spanned by vectors of the form

\[\sum_{a \in H} \psi(a)a(\tilde{e}_i).\]

If \(\varphi_\psi\) is a lift of \(\varphi\) to \(\Gamma_\psi\), then \(C_\psi\) is \(\varphi_\psi\) invariant. Then the action of the lift of \(\varphi\) on the space \(C_\psi\) by multiplication by the matrix \((A_\varphi)_\psi\), and its characteristic polynomial is \(\Delta_\psi\).

It’s possible to view the space \(C_\psi\) somewhat differently (though we won’t use this identification). Then the complex \(C_*(\Gamma_\psi, \mathbb{Z})\), and the space \(\mathbb{C}\) are both
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2.2.2. Fourier transforms on Abelian groups. Before we proceed, we need to recall some standard facts about the Fourier transform \( \mathbb{Z}^n \). Note that in the definitions that follow, all \( L \)-spaces are spaces of functions into \( \mathbb{C} \). Recall that \( T = (S^1)^n \) is the group of characters of \( \mathbb{Z}^n \). Let \( f \in L^1(\mathbb{Z}^n) \). We define a function \( \hat{f} \in L^1(T) \) by setting:

\[
\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^n} f(x) \xi(x)
\]

We summarize the facts we need in the following proposition.

**Proposition 2.4.** The Fourier transform satisfies the following properties.

(a) It extends to an isometric embedding \( L^2(\mathbb{Z}^n) \to L^2(T, \nu) \).

(b) Given \( f, g \in L^1(\mathbb{Z}^n) \), one has that:

\[
\hat{f} \cdot \hat{g} = \hat{f} \cdot \hat{g}
\]

(c) The transform \( L^1(T) \to L^1(\mathbb{Z}^n) \) given by:

\[
f(x) = \int_T \hat{f}(\xi) \xi(x) d\nu(\xi)
\]

where \( \nu \) is the Haar measure on \( T \) is the adjoint of the Fourier transform when restricted to the class of continuous functions, and is an \( L^2 \) isometry on this class.

2.2.3. Passing to abelian covers.

**Proposition 2.5.** Let \( \pi : \Gamma_0 \to \Gamma \) be a characteristic cover to which \( \varphi \) lifts to a continuous map \( \varphi_0 \), and suppose \( (\varphi_0) \), acts trivially on \( H_1(\Gamma_0, \mathbb{Z}) \). Denote \( H_1(\Gamma_0, \mathbb{Z}) \cong \mathbb{Z}^k \), \( K = \# E(\Gamma_0) \) and let \( t_0 = \text{Tr} A_{\varphi_0} \in \mathbb{Z}[\mathbb{Z}^k] \). We can view \( \mathbb{Z}[\mathbb{Z}^k] \) as a subspace of \( L^2[\mathbb{Z}^k] \). If there exists \( \xi \in (S^1)^k \) such that \( |\hat{t}_0(\xi)| > K \), then there is a characteristic cover \( \pi_1 : \Gamma_1 \to \Gamma \) and a lift \( \varphi_1 : \Gamma_1 \to \Gamma_1 \) such that \( (\varphi_1)_* \) acts with infinite order on \( H_1(\Gamma_1, \mathbb{Z}) \).

**Proof.** Note that \( t_0 \) is an element of \( L^2[\mathbb{Z}^k] \) with finite support. If we write \( t_0 = \sum a_i \xi_i \), and take \( \zeta : \mathbb{Z}^k \to \mathbb{C} \) we have that \( \hat{t}_0(\xi) = \sum a_i \xi(\xi_i) \). For any \( \bar{x} \in \mathbb{Z}^k \), the function \( \zeta \to \zeta(\bar{x}) \) is continuous, and thus \( \hat{t}_0 \) is continuous.

Since \( \hat{\xi}_{\varphi_0} \) is a continuous function, we can assume that \( \xi \) is a torsion element of \( (S^1)^k \), and thus corresponds to a a representation of \( \mathbb{Z}^k \) with finite image. Let \( r = \# \xi(\mathbb{Z}^k) \), and let \( \Gamma_1 \) be the cover of \( \Gamma_0 \) associated to the kernel of the
map $H_1(\Gamma_0, \mathbb{Z}) \to H_1(\Gamma_0, \mathbb{Z}/r\mathbb{Z})$. This is a characteristic cover, and thus we can lift $\varphi_0$ to a map $\varphi_1 : \Gamma_1 \to \Gamma_1$.

The matrix $(A_{\varphi_0})_k$ is a $K \times K$ matrix with trace greater than $K$, and thus it has eigenvalues off of the unit circle. We deduce that the action $(\varphi_1)_*$ on $C_1(\Gamma_1, \mathbb{Z})$ has an eigenvalue off the unit circle. The Proposition now follows from the following claim.

**Claim 2.6.** Let $\Gamma$ be a graph with fundamental group $F_n$ and let $\varphi : \Gamma \to \Gamma$ be a continuous function inducing the automorphism $f \in \text{Aut}(F_n)$. Suppose that the action of $\varphi$ on $C_1(\Gamma, \mathbb{C})$ has an eigenvalue with absolute value $> 1$. Then the same is true for the action of $\varphi$ on $H_1(\Gamma, \mathbb{C})$.

To see this, let $U = C_1(\Gamma, \mathbb{C})$, and let $W \subset U$ be the subspace spanned by all closed paths in $\Gamma$. We have a natural identification $W \cong H_1(\Gamma, \mathbb{C})$. The space $U$ is spanned by elements of the form $1_e$, where $e$ ranges over all edges of $\Gamma$. Pick any norm $\| \cdot \|$ on $U$, and let $\lambda > 1$ be the spectral radius of the action of $\varphi$ on $U$. Then there exists an edge $e$ such that

$$\limsup_{k \to \infty} \frac{1}{k} \log \| \varphi^k(1_e) \| = \log \lambda$$

Denote $L_\lambda$ to be the direct sum of all generalized eigenspaces corresponding to eigenvalues with absolute value $\lambda$. Setting $v_k = \frac{\varphi^k(1_e)}{\| \varphi^k(1_e) \|}$, we have that the distance from $v_k$ to $L_\lambda$ goes to 0 as $k \to \infty$. Note that $\varphi^k(1_e)$ corresponds to a path in $\Gamma$, and any such path can be closed to a loop using a bounded number of edges. Thus, the distance of $v_k$ from $W$ goes to 0 as $k \to \infty$. Therefore $L_\lambda \cap W \neq \{0\}$. Since this space is $\varphi$-invariant, it contains an eigenvector with eigenvalue of absolute value $\lambda$.

\[ \square \]

2.3. **Homological shadows and extremal subgraphs.** It is not obvious how to find a $\xi$ that satisfies the conditions of 2.5. Our plan is to find such a $\xi$ by studying the geometry of the support of $t_0 = \text{Tr} A_{\varphi}$, and using properties of the Fourier transform. In this section we study the shape and structure of the support of $t_0$ using a tool called homological shadows.

2.3.1. **Homological shadows of paths.** From now on fix $\Gamma = V_1^n S^1$, a wedge of $n$ circles. Given a path $p$ in $\Gamma$ originating at $\beta$, let $\tilde{p}$ be the lift of $p$ to $\tilde{\Gamma}$ originating at $\tilde{\beta}$. Define the $S(p) \in G$ to be $S(p) = \sum c_i \tilde{e}_i$, where $c_i$ is the number of times $\tilde{p}$ passes through the edge $\tilde{e}_i$ in either direction (so we always have that $c_i \geq 0$). Given an element $w \in \pi_1(\Gamma, \beta)$ let $S(w) = S(\piw)$ where $\piw$ is the path corresponding to a reduced representative of $w$.

Let $V$ be the set of vertices in $\tilde{\Gamma}$. Fix a preferred lift $\tilde{\beta}$ of $\beta$. This gives an identification of $V$ with $H \cong \mathbb{Z}^n$. We have a natural inclusion $H \to H_1(\Gamma, \mathbb{R})$,
which gives us an identification of \( V \) with a subset of \( \mathbb{R}^n \). Define the *shadow of* \( p \) to be \( S(p) = \text{Support}(S(p)) \cap V \subset H \subset H_1(F_n, \mathbb{R}) \). Similarly, for an element \( w \in \pi_1(\Gamma, \beta) \) define the *shadow of* \( w \), or \( S(w) \subset H_1(F_n, \mathbb{R}) \) to be the support of \( S(w) \cap V \).

2.3.2. Homological shadows of \( f \) and \( \varphi \). In [10], the following Theorem is proved.

**Theorem 2.7.** Suppose \( f \) is a train track representative with a Perron-Frobenius transition matrix and \( f^* \) has finite order. There exists a convex polytope \( S_f \subset H_1(F_n, \mathbb{R}) \) with rational vertices such that for any \( w \in F_n \) with infinite \( f \)-orbit,

\[
\lim_{k \to \infty} \frac{1}{k} S_f^k(w) = S_f
\]

where the above limit is taken in the Hausdorff topology. We call \( S_f \) the shadow of \( f \).

Note that while the proof of the above theorem utilizes the existence of a train track graph to calculate \( S_f \), the result holds for any graph with free fundamental group. For the purposes of our proof we require \( \Gamma - \bigvee S^1 \). Also, while we are interested mainly in \( S_f \), for technical reasons we need to consider a different shadow that is attached to \( \varphi \). In the course of proving the above theorem we show the following.

**Proposition 2.8.** Under the above conditions, there exists a convex polytope with rational vertices \( S_\varphi \subset H_1(F_n, \mathbb{R}) \) called the shadow of \( \varphi \) such that for any path \( p \) in \( \Gamma \) with infinite \( \varphi \) orbit,

\[
\lim_{k \to \infty} \frac{1}{k} S_\varphi^k p = S_\varphi
\]

Furthermore, \( S_\varphi \) contains all points of the form \( t_n(\gamma) \) where \( \gamma \) is a cycle in \( T \) and is the convex hull of all elements of the form \( t_n(\gamma) \), where \( \gamma \) is a simple cycle in \( T \).

A priori, the shadows \( S_f \) and \( S_\varphi \) do not have to be equal. They are however calculated in a similar manner. The shadow \( S_f \) is calculated by taking the convex hull of \( t_n(\gamma) \) where \( \gamma \) is a cycle in the train track transition graph of a train track representative of \( f \). If we choose a fixed vertex \( \beta \) in such a graph, and replace \( f \) with an element in its outer automorphism class induced by the automorphism on \( \pi_1(\Gamma, \beta) \) then \( S_\varphi = S_f \) (see [10] for details). In the sequel we will always assume that \( S_\varphi = S_f \).

The convex hull described above was considered by Fried in [7] for the case of an automorphism coming from a pseudo-Anosov diffeomorphism, and in a slightly different language by Dowdall, Kapovich and Leininger ([4], [5]) and separately by Algom-Kfir, Hironaka and Rafi ([1]) for the general Out\((F_n)\) case.
In both cases, this convex hull gives a cross-section of the dual cone to a fibered cone (see the above references for definitions) which is a $n+1$ dimensional cone when $f$ is in the Torelli subgroup. Thus, we have the following.

**Proposition 2.9.** The polytope $Sf$ is $n$ dimensional.

2.3.3. Homological shadows and extremal subgraphs.

**Definition 2.10.** Let $v$ be a vertex of $S\phi$. Let $T_v \subset T$ be the union of all cycles $\gamma$ satisfying $t_n(\gamma) = v$. We call $T_v$ an extremal subgraph.

Extremal subgraphs have nice properties, which are consequences of the following definitions, that are useful in the study of $\text{Tr}(A_{\phi})$.

**Definition 2.11.** Given $x, y \in \mathbb{Z}[H]$ we say that $x$ is subordinate to $y$ and write $x \preceq y$ if $y|\text{Support}(x) = x$. We say that $x$ and $y$ are separated and write $x \parallel y$ if $\text{Support}(x) \cap \text{Support}(y) = \emptyset$.

**Definition 2.12.** Let $T' \subseteq T$ be a subgraph. For any $k$, let $t_k[T'] = \sum s(\gamma)t(\gamma)$ where the sum is taken over all cycles of length $k$ in $T'$. Note that $t_k[T']$ is the trace of $(A_{\phi}[T'])^k$.

**Definition 2.13.** Let $T', T''$ be subgraphs of $T$. We say that $T'$ is subordinate to $T''$, and write $T' \preceq T''$ if for all sufficiently large $k$, $t_k[T'] \preceq t_k[T'']$.

**Definition 2.14.** We say that two subgraphs $T', T''$ of $T$ are separated, and denote $T' \parallel T''$ if for all sufficiently large $k$, $t_k[T'] \parallel t_k[T'']$.

The first thing to note is the following.

**Lemma 2.15.** If $v, w$ are two different vertices of $S\phi$ then $T_v, T_w \preceq T$ and $T_v \parallel T_w$.

*Proof.* The result follows directly from the following claim: let $\gamma \subset \Gamma$ be a loop of length $k$, and let $u$ be a vertex. Then $t(\gamma) = ku$ if $\gamma \subset T_u$.

Indeed, the only if direction of the claim shows that for any $k$, $t_k[T_u] = a_u(ku)$ for some number $a_u$. Applying this to the vertices $v, w$ we get $t_k[T_v] = a_v(kv), t_k[T_w] = a_w(kw)$. Since $v \neq w$ we have by definition that $T_v \parallel T_w$. The coefficient of $kv$ in $t_k[T]$ is the sum of the signs of all loops of length $k$ in $T$ with translation equal to $kv$.

The if direction of the claim shows that this number is equal to the sum of the signs of all loops of length $k$ in $T_v$ with translation equal to $kv$. By definition, this means that $T_v \preceq T$. Similarly, $T_w \preceq T$.

We now turn to prove the claim. The if follows from the definition of $T_u$. To see the only if direction, suppose $\gamma' \subset T_u$ is a cycle of length $k$. Let $\omega \in H^1(F_n, \mathbb{R})$ be a cohomology class whose maximum on $S\phi$ is achieved at $u$, (such a class exists, because $u$ is a vertex of the convex polytope $S\phi$). By maximality, $\omega(t(\gamma')) \leq k\omega(u)$. 


Suppose that \( \omega(t(\gamma')) < k\omega(u) \). Write \( \gamma' = e_1 \ldots e_r \). By definition of \( \mathcal{T}_u \), there are paths \( \eta_1, \ldots, \eta_r \) in \( \mathcal{T}_u \) of lengths \( k_1 - 1, \ldots, k_r - 1 \) such that for every \( i \): \( e_i\eta_i \) is a cycle and \( \omega(t(e_i\eta_i)) = k_i\omega(u) \). Let \( \eta \) be the cycle \( \eta_r \ldots \eta_1 \). Let \( \tau = \omega \circ t \). Then:

\[
\tau(\eta) = \sum_i \tau(\eta_i) = \sum_i [k_i\omega(u) - \tau(e_i)] = \omega(u) \sum_i k_i - \tau(\gamma')
\]

Notice that \( \omega(u) \sum_i k_i = \text{length}(\eta)\omega(u) + k\omega(u) \), and that \( k\omega(u) - \tau(\gamma) > 0 \). Thus, \( \tau(\eta) > \text{length}(\eta)\omega(u) \), which is a contradiction to the definition of \( \omega \). Thus, \( \tau(\gamma') = \text{length}(\gamma')\omega(u) \).

\[\square\]

**Observation 2.16.** The definitions in this subsection, as well as Lemma 2.15 do not use any topological or homological properties. Indeed, they are quite general. They hold whenever we have a directed graph \( \Delta \) and an additive function from the cycles in \( \Delta \) to some finite dimensional rational vector space which doesn’t depend on cyclic reordering of the cycles.

Extremal subgraphs behave quite well with respect to covers. One fact that we need is the following. If \( \mathcal{T}_v, \mathcal{T}_w \) are extremal subgraphs corresponding to different vertices, \( \pi_0 : \Gamma_0 \to \Gamma \) is a finite cover, and \( \mathcal{T}_v', \mathcal{T}_w' \) are the lifts of \( \mathcal{T}_v, \mathcal{T}_w \) respectively, then \( \mathcal{T}_v' \parallel \mathcal{T}_w' \).

**2.4. Stable covers.** Let \( \pi : (\Gamma_0, \beta_0) \to (\Gamma, \beta) \) be a finite characteristic cover to which \( \varphi \) can be lifted to a map \( \varphi_0 \), and let \( v \) be a vertex of \( S\varphi \).

**Definition 2.17.** We say that the vertex \( v \) is **stable in the cover** \( \Gamma_0 \) if the matrix \( A_{\varphi}[\mathcal{T}_v, \pi_0] \) is not nilpotent. We say that the cover \( \Gamma_0 \) is **stable** if all the vertices of \( S\varphi \) are stable in it.

Stable covers are well suited for applying Proposition 2.5. In this section we prove the following.

**Proposition 2.18.** Let \( \pi_0 : \Gamma_0 \to \Gamma \) be a characteristic finite stable cover. Let \( d = \dim H_1(\Gamma_0, \mathbb{Z}) \). If the action of \( \varphi_0 \) on \( H_1(\Gamma_0, \mathbb{Z}) \) has finite order, then there exists an integer \( j \), and an element \( \xi \in (S^1)^d \) such that \( |\xi(s_j)| > \#E(\Gamma_0) \)

where \( s_j = \text{Tr}(A_{\varphi})^j \)

First, we note that by replacing \( f \) with a power of itself we may assume that \( \varphi_0 \) acts trivially on \( H_1(\Gamma_0, \mathbb{Z}) \). Let \( \mathcal{D} \) be the deck group of the cover \( \pi_0 \). We can view \( \mathcal{D} \) and \( f \) as elements of \( \text{Out}(\pi_1(\Gamma_0)) \). Since \( \varphi \) lifts to \( \Gamma_0 \), we have that \( f \) normalizes \( \mathcal{D} \). By replacing it with a power of itself, we can assume that \( f \) centralizes \( \mathcal{D} \). Fixing a vertex \( \beta_0 \in V(\Gamma_0) \), we can identify \( V(\Gamma_0) \) with \( \mathcal{D} \). Since \( f \) centralizes \( \mathcal{D} \), we get that \( \varphi_0 \) acts trivially on \( V(\Gamma_0) \).
This means gives that for any \( \delta \in \mathcal{D} \), the maps \( \delta \circ \varphi_0 \), \( \varphi_0 \circ \delta \) have the same induced action on \( G \). Thus, if we set \( t_0 = \text{Tr} A_\varphi[\pi_0] \), we get that \( t_0 \) is \( \mathcal{D}_* \)-invariant, where \( \mathcal{D}_* \) denotes the induced action of \( \mathcal{D} \) on \( H_1(\Gamma_0, \mathbb{Z}) \).

**Observation 2.19.** A stronger property of \( t_0 \) holds as well. The trace \( t_0 \) is the sum of all the elements along the diagonal of \( A_\varphi[\pi_0] \). The rows of \( A_\varphi[\pi_0] \) correspond to the edges of \( \Gamma_0 \). By picking one edge in each \( \mathcal{D} \) orbit, we get that there exists \( s_0 \in \mathbb{Z}[\mathbb{Z}^d] \) such that \( t_0 = \sum_{\delta \in \mathcal{D}} \delta_*(s_0) \). Thus, for any \( h \in \text{Support}(t_0) \) we get that \( \sum_{\delta \in \mathcal{D}} a_{\delta h} \) is divisible by \( |\mathcal{D}| \), where \( a_{\delta h} \) is the coefficient of \( \delta h \) in \( t_0 \).

**Definition 2.20.** For any matrix \( X \in M_d[\mathbb{Z}^d] \), and any integer \( k \) define the \( k \) trace of \( X \) to be:

\[
\text{Tr}_k[X] = \sum_{h \in kH_1(\Gamma_0, \mathbb{Z})} a_h
\]

where \( a_h \) is the coefficient of \( h \) in \( \text{Tr} X \).

Let \( v \) be a vertex of \( S \varphi \). Pick \( v' \in H_1(\Gamma_0, \mathbb{Q}) \) such that \( (\pi_0)_*v' = v \) (we can pick such a \( v' \) since the vertices of \( S \varphi \) have rational coordinates). Let

\[
\varpi = \sum g \in \mathcal{D} g_*v' \in H_1(\Gamma_0, \mathbb{Q})
\]

be a \( \mathcal{D} \) invariant lift of \( v \) to \( H_1(\Gamma_0, \mathbb{Z}) \). In [10] it is shown that for any \( i \):

\[
S \varphi^i = iS \varphi
\]

Therefore, we can pick an integer \( i \) such that \( \varpi \in H_1(\Gamma_0, \mathbb{Z}) \). Thus, by replacing \( f \) with \( f^i \), we may assume that \( \varpi \in H_1(\Gamma_0, \mathbb{Z}) \).

Denote \( B_v = \varpi^{-1} A_\varphi[T_v, \pi_0] \). The matrix \( B_v \) is a translation of the matrix \( A_\varphi[T_v, \pi_0] \) corresponding the the vertex \( v \), where the translation is introduced to simplify calculations in the sequel. Since \( \pi_0 \) is a stable cover, the vertex \( v \) is stable in \( \pi_0 \). This means that the matrix \( A_\varphi[T_v, \pi_0] \) and hence also the matrix \( B_v \) are not nilpotent. Thus \( \text{Tr}(B_v^i) \neq 0 \) for infinitely many \( i \). We prove the following stronger claim.

**Lemma 2.21.** For all sufficiently large \( k \), there exist infinitely many \( i \) such that \( \text{Tr}_k[B_v^i] \neq 0 \).

**Proof.** Let \( L \) be a lattice in \( \mathbb{Z}^d \), and let \( \Psi_L \) be its characteristic function. Let \( \mathbb{T} = (S^1)^d \). The function \( \Psi_L \) is not in \( L^1 \). We can however still define \( \hat{\Psi}_L \), its Fourier transform as distribution on \( L^1[\mathbb{T}] \). This distribution has the property that given a continuous \( g \in C[\mathbb{T}] \):

\[
\hat{\Psi}_L(g) = \Psi \cdot g
\]

where \( \Psi \cdot g = \sum_{h \in L} a_h \), and \( a_h \) is the coefficient of \( h \) in \( g \).
The distribution $\widehat{\Psi}_L$ is simple to calculate. Indeed, we have that:

$$\widehat{\Psi}_L = \frac{1}{|A_L|} \sum_{\xi \in A_L} \delta_\xi$$

where $A_L = \{ \xi : \xi|_L = 1 \}$, and $\delta_\xi$ is the delta distribution at $\xi$.

This is particularly simple to see when $g$ is a polynomial. Suppose $g = \sum a_h h$. Then:

$$\frac{1}{|A_L|} \sum_{\xi \in A_L} \delta_\xi(g) = \frac{1}{|A_L|} \sum_{\xi \in A_L} \delta_\xi(\sum_h a_h h) = \frac{1}{|A_L|} \sum_h a_h \sum_{\xi \in A_L} \hat{h}(\xi)$$

The set $A_L$ is the set of all homomorphisms from $H_1(\Gamma_0, \mathbb{Z}) \to \mathbb{C}$ whose kernel contains $L$. All such homomorphisms have finite image. For any $h$, the sum $\hat{h}(\xi)$ is the repeated sum of all $j$-th roots of unity for some $j$. For $h \in L$, we have $j = 1$, and by definition of $A_L$ that $\sum_{\xi \in A_L} \hat{h}(\xi) = |A_L|$. For $h \notin L$ we have $j \neq 1$ and thus $\sum_{\xi \in A_L} \hat{h}(\xi) = 0$. Thus, $\frac{1}{|A_L|} \sum_{\xi \in A_L} \delta_\xi(g) = \sum_{h \in L} a_h = \Psi_L \cdot g$.

Let $\widehat{B}_v \in M_d[L^2(T)]$ be the Fourier transform of $B_v$, here the Fourier transform is taken entry by entry. The entries of $\widehat{B}_v$ are then functions from $(S^1)^d$ to $\mathbb{C}$. For any $\xi \in (S^1)^d$, we can form the matrix $\widehat{B}_v[\xi] \in M_d(\mathbb{C})$ by plugging $\xi$ into each coordinate of $\widehat{B}_v$. We call this matrix the specialization of $\widehat{B}_v$ at $\xi$.

Let $\mathcal{X} \subset (S^1)^d \times \mathbb{C}^d$ be the set of all $(\xi, z_1, \ldots, z_d)$ such that $(z_1, \ldots, z_d)$ are the eigenvalues of $\widehat{B}_v[\xi]$, with multiplicity. Let $P : \mathcal{X} \to (S^1)^d$ be the projection $P(\xi, z) = \xi$. The map $P$ is a local homeomorphism. A root is a function $\rho : \mathbb{T} \to \mathbb{C}^d$, $\rho = (\rho_1, \ldots, \rho_d)$ such that $(\text{Id}, \rho) : \mathbb{T} \to \mathbb{T} \times \mathbb{C}^d$ is a section of the projection $P$. More explicitly, $\rho$ is a root if for every $\xi \in \mathbb{T}$, $(\rho_1(\xi), \ldots, \rho_d(\xi))$ is the collection of eigenvalues of $\widehat{B}_v[\xi]$ counted with multiplicity.

Since $P$ is a local diffeomorphism roots always exist, but they may not always be continuous. However, if $Y \subset \mathbb{T}$ is a contractible subset, then we can always find a root that restricts to a continuous function from $Y$ to $\mathbb{C}^d$. Let $Z$ be a finite collection of hyperplanes of hyperplanes in $\mathbb{T}$ such that $Y = \mathbb{T} \setminus Z$ is contractible (here a hyperplane is the image in $\mathbb{T}$ of an affine hyperplane in $\mathbb{R}^d$ under the covering space map $\mathbb{R}^d \to \mathbb{T}$). Without loss of generality, we can assume that $Z$ contains no rational points in $\mathbb{T}$.

Pick a root $\rho = (\rho_1, \ldots, \rho_d)$ that is continuous on $Y$. Notice that for any number $j$, and any $\xi \in \mathbb{T}$ we have that

$$\text{Tr} \widehat{B}_v^j(\xi) = \sum_{i=1}^d \rho_i^j(\xi).$$
Fix an integer $k$, and let $L = k\mathbb{Z}^d$. We would like to use the inverse Fourier transform to study $\text{Tr}_k(B_v)$ using the functions $\rho_1, \ldots, \rho_d$ and the distribution $\widehat{\Psi}_L$. Unfortunately, the isometry of the inverse Fourier transform does not hold for non-continuous functions. We address this issue in the following way.

Let $U$ be a small neighborhood of $Z$ (we take $U$ to be small enough so that it does not intersect $A_L$.) We replace $\rho_1, \ldots, \rho_d$ with continuous functions $\rho_1', \ldots, \rho_d'$ such that for any $i$, $\rho_i = \rho_i'$ on $Y$, and $\rho_1' + \ldots + \rho_d' = \widehat{\text{Tr}[B_v]}$. Because $\rho_1' + \ldots + \rho_d' = \widehat{\text{Tr}[B_v]}$, and the $\rho_i'$ are continuous, we have that $\sum \widehat{\Psi}_L(\rho_i') = \text{Tr}_k(B_v)$. Since $U$ does not intersect $A_L$, we have that $\text{Tr}_k[B_v] = \sum \widehat{\Psi}_L(\rho_i')$. Similarly, for any integer $j$ we have that:

$$\text{Tr}_k(B_v^j) = \frac{1}{|A_L|} \sum_{i=1}^{d} \sum_{\xi \in A_L} \rho_i(\xi)^j$$

By the Vandermonde theorem, this expression is non-zero for infinitely many $j$ if and only if $\rho_i|_{A_L} \neq 0$ for some $i$. Since $\rho_i$ is continuous on an open, dense subset of $T$, and the sets $A_L = A_L(k)$ become equi-distributed as $k \to \infty$, we get that for all sufficiently large $k$, $\rho_i|_{A_L} \neq 0$ for every $i$. The result now follows.

**Lemma 2.22.** There exist infinitely many $k$ such that $\text{Tr}_k(B_v) > 0$ for every vertex $v$ of $S\varphi$.

**Proof.** Pick a sufficiently large $k$ which satisfies the conclusion of Lemma 2.21 for all vertices of $S\varphi$. As in the previous lemma, let $L = k\mathbb{Z}^d$. Let $v$ be a vertex, and let $\rho = (\rho_1, \ldots, \rho_d)$ be the root constructed in the previous lemma with respect to the vertex $v$.

Let $\mu_v = \max\{|\rho_i(\xi)| : 1 \leq i \leq d, \xi \in A_L\}$. Let $X_v = \{(i, \xi) : |\rho_i(\xi)| = \mu_v\}$, and $l_v = |X_v|$. Choose some arbitrary order on the elements of $X_v$ and let $\alpha_v$ be the vector $\alpha_v = \left(\frac{\rho_i(\xi)}{\mu_v}\right)_{(i, \xi) \in X_v}$.

Let $s$ be the number of vertices of $S\varphi$. Choose some arbitrary order on the vertices of $S\varphi$. Let $l = \sum l_v$, and let $\alpha$ be the vector $\alpha = (\alpha_v)_v \in (S^1)^s$. Define a function $F : (S^1)^s \to \mathbb{C}^s$, by setting the $i$th coordinate of $F(\xi_1, \ldots, \xi_s)$ to be the sum of all the $\xi_j$ corresponding to the $i$th vertex of $S\varphi$.

If $\alpha$ is a rational point, then there is some power $j$ of $\alpha$ such that $\alpha^j = (1, \ldots, 1)$. Notice that $F$ of this point is positive in all of its coordinates. By replacing $j$ with a sufficiently large multiple of itself, we get that $\widehat{\Psi}_L(\text{Tr}_k[B_v^j]) > 0$, for all $v$. If $\alpha$ is not a rational point then, after replacing $\alpha$ with some power of itself we get that the set $\{\alpha^i\}_{i=1}^{\infty}$ is equidistributed in some sub-torus $T' \subset (S^1)^s$ that does contain a rational point and is closed under multiplication.

The function $F$ is continuous on $T'$. Since it contains a rational point, there is an open subset $V$ of $T'$ where all the coordinates of $F$ have real parts that
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are greater than $\frac{1}{2}$. Since the orbit of $\alpha$ is equidistributed in $T'$, there are infinitely many $j$ such that all coordinates of $F(\alpha^j)$ have real parts that are greater than $\frac{1}{2}$. For all sufficiently large $j$ of this form, and for any vertex $v$ we get that $\tilde{\Psi}_L(\text{Tr}_k[B^j]) > 0$.

□

Suppose $h \in \mathbb{Z}^d$. Notice that it is enough to prove Proposition 2.18 for the matrix $hA_\varphi[\pi_0]$ as opposed to $A_\varphi$, because multiplication by $h$ acts by isometries on $L^2[\mathbb{Z}^d]$. Thus, we may assume that 0 is a vertex of $S_\varphi$.

**Lemma 2.23.** There exists a lattice $L \subset \mathbb{Z}^n$ such that $S_\varphi \cap L \subset V(S_\varphi)$, and $|L \cap S_\varphi| > n$, where $V(S_\varphi)$ is the set of vertices of $S_\varphi$.

**Proof.** Denote $P = S_\varphi$. Let $F$ be a face of $P$. We say that a face $F'$ of $P$ opposite from $F$ if there exists $\alpha \in H^1(\Gamma; \mathbb{R})$ that takes its minimum value in $P$ precisely on $F$ and its maximum value on $F'$. Let $F$ be a co-dimension 1 face of $P$ that has 0 as a vertex. Let $F'$ be an opposite face such that there exists $\alpha \in H^1(\Gamma_0, \mathbb{R})$ that takes on its minimal value of 0 precisely at $F$ and takes on its maximum value precisely at $F'$.

We show the slightly stronger claim that there is a lattice $L$ whose intersection with $P$ consists solely of vertices, has at least $n+1$ vertices, and whose intersection with $P$ is contained in $F \cup F'$. We prove this by induction on $n$. For $n = 1$, the result is obvious. Suppose we have proved it up to $n$, and wish to prove it for $n+1$.

Denote $A = V(F)$, $B = V(F')$. For $v \in B$, set $B_v = \{v' - v | v' \in B\} \subset \text{span}(F)$, and let $Q_v$ be the convex hull of $A$ and $B_v$. It is possible to pick $v$ such that 0 is a vertex of $Q_v$. Indeed, we can pick $\alpha \in H^1(\Gamma_0, \mathbb{R})$ whose minimum on $F$ is achieved at 0 and whose minimum on $F'$ is achieved at precisely one vertex. Take this vertex to be $v$.

Let $L'$ be the lattice associated to $Q_v$ and any co-dimension 1 face of $Q_v$ by the induction hypothesis, and let $L = \langle L', v \rangle$. By definition, the lattice $L$ intersects $P$ only at vertices which are contained in $F \cup F'$. Furthermore, this intersection includes 0, $v$ and at least $d-1$ other vertices. If this lattice is not $n$-dimensional, we can extend it by adding further linearly independent vectors without increasing its intersection with $P$. Thus, this lattice satisfies the conditions of the lemma.

□

We are now able to prove Proposition 2.18.

**Proof.** Pick numbers $k, j$ satisfying the conclusion of Lemma 2.22. Let $L'$ be a lattice in $H^1(\Gamma, \mathbb{Z})$ satisfying the condition of Lemma 2.23. Let $\{v_1, \ldots, v_r\} = L' \cap S_\varphi$. Let $\overline{v}_1, \ldots, \overline{v}_r$ be $D$-invariant lifts of these vector to $H^1(\Gamma_0, \mathbb{R})$. For each $1 \leq i \leq r$, let $L_i = \text{Tr}[B^j_{\overline{v}_i}] \cap k\mathbb{Z}^d$. Consider the lattice $\langle L_1, \ldots, L_r \rangle$. We
can add additional vectors to this lattice without increasing its intersection with Support $\text{Tr} A_{\varphi}^j[\pi_0]$ so that it is a $d$ dimensional lattice. Call the resulting lattice $L$.

Let $\Psi_L$ be its characteristic function. We have that

$$
\Psi_L \cdot \text{Tr} A_{\varphi}^j[\pi_0] = \sum_{i=1}^r \text{Tr}_k[B_{\psi_i}^j]
$$

Note that $\text{Tr}_k[B_{\psi_i}^j] > 0$ for every $i$. Furthermore, since the set $\pi + k\mathbb{Z}^d$ is $\mathcal{D}$ invariant, we have by observation 2.19 that $\text{Tr}_k[B_{\psi_i}^j]$ is divisible by $|\mathcal{D}|$. Thus

$$
\Psi_L \cdot \text{Tr} A_{\varphi}^j[\pi_0] \geq (n+1)|\mathcal{D}| > n|\mathcal{D}| = \#E(\Gamma_0)
$$

Since $\Psi_L \cdot \text{Tr} A_{\varphi}^j[\pi_0] = \frac{1}{|A_L|} \sum_{\xi \in A_L} \text{Tr} A_{\varphi}^j[\pi_0](\xi)$, we get that there exists $\xi \in A_L$ such that $|\text{Tr} A_{\varphi}^j[\pi_0](\xi)| > \#E(\Gamma_0)$, as required. \(\square\)

### 2.5. Stabilizing vertex subgraphs.

We are now left with the problem of finding stable covers of $\Gamma$. We address this problem by finding, for each vertex $v$ of $\mathcal{S}_\varphi$ a cover where it is stable.

**Observation 2.24.** If $T_v$ is the vertex subgraph corresponding to the stable vertex $v$, and $T'_v$ is the lift of $T_v$ to some cover $\pi$ where $\varphi$ lifts and acts trivially on first homology, then $T'_v$ is also stable (that is - $\text{Tr} A_{\varphi}^j[T_v, \pi]$ is not nilpotent). If we can find for each vertex $v$ a cover $\pi_v$ where it is stable then by taking a common refinement we can find a cover $\pi$ where $f_*$ has infinite order, or all lifts of all vertices are stable. If all of the covers in question are characteristic, then so is $\pi$.

Our goal in this section is to prove the following proposition.

**Proposition 2.25.** For any vertex $v$ of $\mathcal{S}_\varphi$, there exists a finite characteristic cover $\pi_0 : \Gamma_0 \to \Gamma$ such that the lift of $T_v$ in $\Gamma_0$ is stable, or a finite characteristic cover where the homological action of a lift of $\varphi$ has infinite order.

We begin by giving some definitions, and recalling standard facts about nilpotent groups.

### 2.5.1. Assigning subgroups to subgraphs.

Let $\Pi = \eta_1 \ldots \eta_k$ be a path of length $k$ in $\mathcal{T}$. This path corresponds to a subpath of $\varphi^k(\eta_1)$ in $\Gamma$ whose last element is $\eta_k^{\pm 1}$. Write this path as $g = r\eta_k^{\pm 1}$. This subpath is a loop in $\Gamma$, and thus gives a word in $F_n$.

**Definition 2.26.** Define the group element of $\Pi$ or $g(\pi)$ to be the element of $F_n$ corresponding to $r$ if $g = r\eta_k$ and $g$ otherwise.

The word $g(\Pi)$ has the property that its image in $H_1(F_n, \mathbb{R})$ is $t(\Pi)$. 

Definition 2.27. Let $\mathcal{C}$ be a finite collection of cycles in $\mathcal{T}$. Define $g(\mathcal{C}) = \langle g(\gamma) | \gamma \in \mathcal{C} \rangle \leq \pi_1(\Gamma, \beta)$.

Let $\gamma_1$, $\gamma_2$ be loops based at $\nu$. Suppose $\gamma_1$ corresponds to the sub-path $w\nu$, and $\gamma_2$ corresponds to the sub-path $w\nu^{-1}$. In this case we have $g(\gamma_1) = g(\gamma_2) = w$. The difference between them is that $s(\gamma_1)$ is positive, and $s(\gamma_2)$ is negative. Intuitively, the two loops, $\gamma_1, \gamma_2$ correspond to a an unavoidable cancellation in the free group of part of the word $\phi^k(\nu)$.

For every $\nu \in V(\mathcal{T})$, and any $g \in g(\mathcal{C})$, define $\mathcal{C}(\nu, g, +)$ to be the set of all $\gamma \in \mathcal{C}$ originating at $\nu$ such that $s(\gamma) = 1$, $g(\gamma) = g$. Define $\mathcal{C}(\nu, g, -)$ to be the set of all $\gamma \in \mathcal{C}$ originating at $\nu$ such that $s(\gamma) = -1$, $g(\gamma) = g$.

Definition 2.28. We say that $g(\mathcal{C})$ is non degenerate if there exist $\nu, g$ such that $\# \mathcal{C}(\nu, g, +) \neq \# \mathcal{C}(\nu, g, -)$.

If $v$ is a vertex of $\mathcal{S}f = \mathcal{S}\phi$, and $\mathcal{C}_v$ is the collection of simple cycles in $\mathcal{T}_v$, then $g(\mathcal{C}_v)$ is non-degenerate, since otherwise $v$ would not be a vertex of $\mathcal{S}f$.

2.5.2. Nilpotent quotients of free groups. Let $F$ be a group. The lower central series of $F$ is the sequence of subgroups given by the recursive definition $F_0 = F$, $F_{i+1} = [F, F_i]$. We denote by $N_i = F/F_i$. Let $L_i = F_{i-1}/F_i$. Let $L$ be a finitely generated free group. We require the following standard facts:

(a) The groups $F_i$ are characteristic subgroups of $F$.
(b) The groups $N_i$ are nilpotent.
(c) The groups $L_i$ are finitely generated torsion free abelian groups, and furthermore $L_i = Z(N_i)$. If $S$ is a generating set for $F$, then the set of elements of the form $[a_1, \ldots, a_{i+1}]$ (where $a_1, \ldots, a_{i+1} \in S$, and $[a_1, \ldots, a_{i+1}] = [a_1, [a_2, [a_3, \ldots]]]$) generate $L_i$.
(d) Let $f \in \text{Aut}(F)$ be an automorphism that acts trivially on $L_1$. Then $f$ acts trivially on $L_i$ for every $i$.

2.5.3. Nilpotent-stable covers. Let $v$ be a vertex of $\mathcal{S}\phi$, and let $\mathcal{T}_v$ be its vertex subgraph. For any $i \geq 0$, let $p_i : \pi_1(\Gamma, \beta) \to N_i$ be the quotient map onto the $i^{th}$ nilpotent quotient of $F_n$.

Definition 2.29. For any $k \geq 0$, let $t_k[\mathcal{T}_v, N_i] = \sum s(\gamma) p_i \circ g(\gamma) \in \mathbb{Z}[N_i]$, where the sum is taken over all cycles of length $k$ in $\mathcal{T}_v$. We say that $\mathcal{T}_v$ is $i^{th}$ level nilpotent stable if $t_k[\mathcal{T}_v, N_i] \neq 0$ for infinitely many $k$.

Lemma 2.30. There exists an $i$ such that $\mathcal{T}_v$ is $i^{th}$ level nilpotent stable.

Proof. Since $\mathcal{T}_v$ is a vertex graph, for any two cycles $\gamma_1, \gamma_2$ of length $k$ in $\mathcal{T}_v$ we have that $p_1 \circ g(\gamma_1) = p_1 \circ g(\gamma_2) = kv$. Pick $x \in F_n$ whose image in $H$ is $v$. For any cycle $\gamma$ of length $k$ in $\mathcal{T}_v$, we have that $p_2(x^{-1}g(\gamma)) \in L_2$. 

Since $f$ acts trivially on $H_1(F_n, \mathbb{Z})$, for any cycle $\gamma$ in $\mathcal{T}_v$ of length $k$, we can write

$$p_2 \circ \mathfrak{g}(\gamma) = t^{(2)}(\gamma) \prod_{j=0}^{k-1} p_2 \circ f^j(x)$$

for some $t^{(2)}(\gamma) \in L_2$.

Given two cycles, $\gamma_1, \gamma_2$ of lengths $l_1, l_2$ based at the same point, we have that $f^{l_1}(\mathfrak{g}(\gamma_1)) = \mathfrak{g}(\gamma_2)$. Since $L_2$ is central in $N_2$, and $f$ acts trivially on $L_2$, we get $t^{(2)}(\gamma_1 \gamma_2) = t^{(2)}(\gamma_1) t^{(2)}(\gamma_2)$. Similarly, for any cycle $\gamma$ in $\mathcal{T}_v$ of length $k$ that is a concatenation of cycles $\gamma_1, \ldots, \gamma_r$ all based at that same point:

$$t^{(2)}(\gamma) = \prod_{i=1}^{k} t^{(2)}(\gamma_i)$$

Set

$$t^{(2)}_k[\mathcal{T}_v] = \sum \mathfrak{g}(\gamma) t^{(2)}(\gamma) \in \mathbb{Z}[L_2]$$

where the sum is taken over all based cycles of length $k$ in $\mathcal{T}_v$. We then have that:

$$t_k[\mathcal{T}_v, N_1] = t^{(2)}_k[\mathcal{T}_v] \prod_{j=0}^{k-1} p_2 \circ f^j(x)$$

In contrast to the expressions of the form $t(\gamma)$ that we used in previous sections, the function $t^{(2)}(\gamma)$ is not necessarily invariant under a cyclic reordering of $\gamma$. However, by the additivity property discussed above, it is invariant under cyclic reorderings that preserve the basepoint.

We define a new function $\text{bft}^{(2)}$ (or basepoint free translation) that assigns to each cycle $\gamma$ an element of $L^V(\mathcal{T}_v)$ in the following way: for a vertex $\nu$ of $\mathcal{T}_v$ and a cycle $\gamma$, define $\text{bft}^{(2)}(\gamma)(\nu)$ to be 0 if $\gamma$ does not pass through $\nu$ and $t^{(2)}(\gamma_{\nu})$ otherwise, where $\gamma_{\nu}$ is a cyclic reordering of $\gamma$ so that it is based at $\nu$.

The function $\text{bft}^{(2)}$ is an additive function from cycles in $\mathcal{T}_v$ to a finitely generated torsion free abelian group whose value does not depend on cyclic reorderings that preserve the basepoint.

As in explained in Observation 2.16 we can define extremal subgraphs in the same way as in section 2.3.3. Thus, we have a (not necessarily proper) vertex subgraph $\mathcal{T}_v^{(2)} \subseteq \mathcal{T}_v$ such that for any cycle $\gamma$ in $\mathcal{T}_v^{(2)}$ of length $k$, $\text{bft}^{(2)}(\gamma) = k v^{(2)}$ for some $v^{(2)} \in L^V(\mathcal{T}_v) \otimes \mathbb{Q}$. Furthermore, if $\mathcal{T}_v^{(2)}$ is $i$th level nilpotent stable then so is $\mathcal{T}_v$.

Let $l$ be the minimal integer such that $l v^{(2)} \in L^V(\mathcal{T}_v)$. By definition of $\mathcal{T}_v^{(2)}$ we have that $k v^{(2)} \in L^V(\mathcal{T}_v)$ whenever $k$ is the length of a cycle in $\mathcal{T}_v^{(2)}$. We get that $k$ is divisible by $l$ whenever $k$ is the length of a cycle in $\mathcal{T}_v^{(2)}$. 
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Using the same reasoning as in the previous paragraph, we can write for any simple loop \( \gamma \) in \( T_v^{(2)} \):

\[
p_3 \circ g(\gamma) = t^{(3)}(\gamma) \prod_{j=0}^{k/l-1} p_3(f_{jl}(x))
\]

for some \( t^{(3)}(\gamma) \in L_3 \). Proceeding analogously to the above discussion, we find a (not necessarily proper) extremal subgraph \( T_v^{(3)} \subseteq T_v^{(2)} \) where for any loop \( \gamma \), \( bft^{(3)}(\gamma) = kv^{(3)} \) where \( k \) is the length of \( \gamma \) and \( v^{(3)} \in L_V^V(T_v^{(2)}) \otimes \mathbb{Q} \). By construction, if \( T_v^{(3)} \) is \( i^{th} \) level nilpotent stable then so is \( T_v^{(2)} \).

Proceeding in this manner, we get a sequence of graphs \( \ldots \subseteq T_v^{(4)} \subseteq T_v^{(3)} \subseteq T_v^{(2)} \subseteq T_v \). Since \( F_n \) is residually nilpotent, there exists an \( i \) such that the subgroup of \( F_n \) generated by the group elements assigned to all the cycles in \( T_v^{(i)} \) based at any given vertex of this graph is cyclic. Indeed, if there were always two such cycles whose group elements did not form a cyclic group then by residual nilpotence one could pass to a further proper subgraph.

Since \( T_v \) is a finite graph, this process has to terminate. Since \( v \) is a persistent vertex, we have that \( T_v^{(i)} \) is \( i^{th} \) level nilpotent stable.

\[\square\]

Lemma 2.31. Suppose that \( T_v \) is the extremal graph of the vertex \( v \), and that \( T_v \) is \( i^{th} \) level nilpotent stable. Then there is a characteristic cover \( \pi_0 : \Gamma_0 \to \Gamma \) where the lift of \( T_v \) is \((i-1)^{th}\) level nilpotent stable, or a characteristic cover where the action of a lift of \( \varphi \) has infinite order action on first homology.

**Proof.** Let \( a_1, \ldots, a_i \in F_n \), and let \( p \in \mathbb{N} \). Explicit calculation shows that

\[
[a_1^p, \ldots, a_i^p] \equiv p^i[a_1, \ldots, a_i] \pmod{F_n^{(i+1)}}
\]

(see for example [3]). Thus, if we set \( K^p < F_n \) to be the kernel of the map \( F_n \to H_1(F_n, \mathbb{Z}/p\mathbb{Z}) \), then the image of \( K^p_{i-1} \) in \( L_{i-1} \) is \( p^{i-1}L_{i-1} \). Note that this cover is characteristic.

Using the notation of Lemma 2.30 let

\[
bft_k^{(i)}[T_v^{(i)}] = \sum s(\gamma) bft_k^{(i)}(\gamma)
\]

where the sum is taken over all loops of length \( k \) in \( T_v^{(i)} \). Let \( bft_k^{(i)}[T_v^{(i)}, p] \) be the sum we get by taking only the summands of \( bft_k^{(i)}[T_v^{(i)}] \) that lie in \( p^{i-1}L_{i-1}^V(T_v^{(i)}) \).

If we show that there exists a \( p \) such that for infinitely many values of \( k \), \( bft_k^{(i)}[T_v^{(i)}, p] \neq 0 \), then we will have that \( T_v \) is \((i-1)^{th}\) level nilpotent stable in \( K_p \).
Denote $M = L_{i-1}^{V(T_v^{(i)})}$. By construction, the map $bft^{(i)}$ gives a homomorphism from $H_1(T_v^{(i)}, \mathbb{Q})$ to $M$. It can thus be extended to a homomorphism from $C_1(T_v^{(i)}) \to M$. Thus, if $T_v^{(i)}$ has $m$ vertices, we can construct a matrix $A \in GL_m(\mathbb{Z}[M])$ such that $t_k^{(i)}[T_v^{(i)}] = \text{Tr} A^k$, for any $k$. We are tasked with finding $p$ such that for infinitely many $k$, $\text{Tr}_p[A^k] \neq 0$ (where $\text{Tr}_p$ is the notation used in the proof of Proposition 2.18).

In Lemma 2.21 we showed that given a non-nilpotent $m \times m$ matrix $A$ with coefficients in $\mathbb{Z}[\mathbb{Z}^d]$ there exists a $p$ such that $\text{Tr}_p(A^k) \neq 0$ for infinitely many values of $k$ (indeed, this is true for all but finitely many values of $p$). Applying this Lemma gives the result.

\[ \square \]

2.5.4. **Concluding the proof.** We now proceed to prove Proposition 2.25.

**Proof.** By Lemma 2.30 the graph $T_v$ is $i^{th}$ level nilpotent stable for some $i$. By repeated application of Lemma 2.31, we can either find a characteristic cover where the homological action has infinite order, or one where the lift of $T_v$ is $1^{st}$ level nilpotent stable. Since this is the same as our original definition of being stable, we are done.

\[ \square \]

2.6. **Proof of Theorem 2.1**

**Proof.** If $f_*$ has infinite order, we are done. Otherwise, by replacing $f$ with a power of itself we can assume that $f$ is in the Torelli group. By Proposition 2.25, for each such vertex we can find a cover where it is stable. By performing this process for every vertex, we find a stable cover of $\Gamma$. Note that we have only taken characteristic abelian covers at each step, so this cover is solvable. Now apply Propositions 2.18 and 2.5 to get the result.

\[ \square \]

2.7. **Proof of Theorem 1.3** Suppose first that $\phi \in \text{Mod}(\Sigma)$ is a multitwist about the multicurve $C$. Add curves to $C$ until it forms a pants decomposition $\mathcal{P}$ of $\Sigma$. Assume first that there are two pants $P_1, P_2 \in \mathcal{P}$ such that $S = P_1 \cup P_2$ is a four holed sphere in $\Sigma$, whose interior contains the curve $\gamma \in C$. Pick a point $p \in S$, and let $\Gamma$ be the image of $\pi_1(S, p)$ in $\pi_1(\Sigma, p)$ under the inclusion homomorphism.

By a theorem of Marshall Hall ([11]), we can find a finite index subgroup $K \subset \pi_1(\Sigma, p)$ such that $\Gamma < K$, and $\Gamma$ is a free factor in $K$. There is a power $\phi^k$ of $\phi$ that lifts to $\gamma$, and whose restriction to a lift of $S$ is a Dehn twist about a lift of the curve $\gamma$. Such a Dehn twist has an infinite order action on the homology of $S$, and hence on the homology of $H_1(K, \mathbb{Z})$ (since $\Gamma$ is a free factor in $K$).
Pick a finite, characteristic cover \( K' < K \). We have an equivariant transfer map \( T : H_1(K, \mathbb{Z}) \to H_1(K', \mathbb{Z}) \). The map \( \phi \) lifts to the cover corresponding to \( K \). By equivariance, some power of \( \phi \) acts with infinite order on \( H_1(K', \mathbb{Z}) \), and hence so does \( \phi \).

Now, we return to the case where we cannot find \( S \) as required. In this case, we can find \( P_1, P_2 \in \mathcal{P} \) such that \( P_1 \cup P_2 \) is either a surface of genus 2, or a two holed torus. Say \( S \) is a two holed torus (the case of a genus two surface is nearly identical). The interior of this torus contains two disjoint simple closed curves \( \gamma_1, \gamma_2 \) that were used in the pants decomposition \( \mathcal{P} \). Assume that \( \gamma_1 \in \mathcal{C} \).

Once again, let \( \Gamma \) be the image of \( \pi_1(S, p) \) in \( \pi_1(\Sigma, p) \) under the inclusion homomorphism, and let \( K \leq \pi_1(\Sigma, p) \) be such that \( \Gamma < K \), and \( \Gamma \) is a free factor in \( K \). We have a homeomorphic lift of \( S \) in the cover corresponding to \( K \), together with lifts of \( \gamma_1, \gamma_2 \). Pick a curve \( \delta \) in this lift intersects \( \gamma_2 \) once, and does not intersect \( \gamma_1 \). We have a surjection \( K \to \mathbb{Z}/2\mathbb{Z} \) sending the curve \( \delta \) to 1. Let \( K_1 \) be the kernel of this surjection. Lift the curves forming the pants decomposition \( \mathcal{P} \) to a multicurve in the cover corresponding to \( K_1 \), and extend the resulting multicurve to a pants decomposition. This pants decomposition can be taken to have two adjacent pants that form a four holed sphere, as above.

This proves the Theorem for a multitwist. By the Nielsen-Thurston classification of mapping class groups, it remains to prove the result for a mapping class \( \phi \) such that \( \phi^k \) restricts to a pseudo-Anosov diffeomorphism of a subsurface \( S \subset \Sigma \) and some \( k \). Let \( F \) be the image of \( \pi_1(S) \) in \( \pi_1(\Sigma) \). By picking a base point \( p \in S \), \( \phi \) induces an element of \( \text{Aut}(\pi_1(\Sigma)) \) that fixes \( F \). By [11], there exists a finite index \( K \leq \pi_1(\Sigma) \) such that \( F \leq K \) and \( F \) is a free sub-factor of \( K \). By passing to a power of \( \phi \) we may assume that it lifts to this cover. Thus, we may assume that \( \phi \) fixes a subsurface of \( \Sigma \) whose fundamental group is a free sub-factor. In this case the image of \( H_1(S, \mathbb{Z}) \subset H_1(\Sigma, \mathbb{Z}) \) is a direct summand.

Let \( \varphi \in \text{Out}(F) \) be the element induced by \( \phi \). To prove Theorem 1.3, it is enough to prove that there is a representative \( f \) of \( \varphi \) in \( \text{Aut}(F) \) and a finite index \( f \)-invariant subgroup \( L \leq F \) such that \( f_* \in \text{GL}(H_1(L, \mathbb{R})) \) has infinite order. To prove this, we proceed in a similar fashion to the proof of Theorem 1.4.

While the mapping class \( \phi \) is a pseudo-Anosov mapping class on \( S \), it may not be one on \( \overline{S} \subset \Sigma \). For instance, \( S \) may have two different boundary components that are identified in \( \overline{S} \), producing a simple curve fixed by \( \phi \). Let \( \mathcal{C} \) be the multi-curve \( \partial S \subset \overline{S} \). By passing to a power of \( \phi \) and possibly restricting to a subsurface, we can assume that \( \mathcal{C} \) is a a canonical reduction system for \( \phi \) in \( \overline{S} \) (that is, the curves in \( \mathcal{C} \) are fixed by \( \phi \), and no other curve in \( \overline{S} \).
has finite φ orbit). If the map φ induces an infinite order element of \( H_1(S, \mathbb{R}) \), we are done. Otherwise, by replacing it with a power we may assume that it acts trivially on \( H_1(S, \mathbb{R}) \).

In our proof of Theorem 2.1 we used the fact that the mapping class in question was pseudo-Anosov in one place only - to show that the convex hull of the vertices of \( S \varphi \) is a full dimensional convex polytope. We use a similar idea in this proof.

Let \( M_\phi \) be the mapping torus of \( \phi \) over \( S \). That is, \( M_\phi = S \times I / \sim \) where \((x, 0) \sim (\phi(x), 1)\). The manifold \( M_\phi \) is a 3-manifold with toroidal boundary components. This manifold comes equipped with a flow \( \mathcal{F} \) given by projecting the flow \( \tilde{\mathcal{F}}_s(x, t) = (x, t + s) \) on \( S \times \mathbb{R} \). Let \( N_\phi \) be the mapping torus of \( \phi \) over \( S \), and let \( \mathcal{F}' \) be the flow on this manifold.

Given \( x \in S \), and \( t \in \mathbb{N} \) we can build a curve in \( M_\phi \) in the following way. Flow \( t \) upwards from \((x, 0)\). Close the resulting ray to a curve by adding the shortest possible path in \( S \times 0 \). Call the resulting curve \( \gamma_t \). Let \([\gamma_t]\) be its image in \( H_1(M_\phi, \mathbb{R}) \). The cone on the set of all accumulation points of \( \frac{1}{t}[\gamma_t] \) as \( t \to \infty \) is called the cone of homological directions. Denote this cone by \( \mathcal{H} \). Similarly, let \( \mathcal{H}' \) be the cone of homological directions in \( N_\phi \). In [7], Fried defined the cones of homological directions, and describes how to calculate them from the first return map of the flow to a cross section.

In our case, by picking a graph \( \Gamma \subset S \) that carries \( \pi_1(S) \) and flowing upwards from every point in the graph, we see that the cone \( \mathcal{H}' \) is a cone with cross section \( S \varphi \). Let \( i : S \to \overline{S} \) be the inclusion map. The map \( i \) induces a surjective map \( i_* : H_1(N_\phi, \mathbb{R}) \to H_1(M_\phi, \mathbb{R}) \). Since any upwards flow path in \( M_\phi \) corresponds to an upward flow path in \( N_\phi \), we get that \( \mathcal{H} = i_*\mathcal{H}' \). Thus, \( \mathcal{H} \) is an open cone in \( H_1(M_\phi, \mathbb{R}) \).

The automorphism \( f \in \text{Aut}(\pi_1(S)) \) induces an automorphism \( g \in \text{Aut}(\overline{S}) \). Let \( \psi \) be a map from a wedge of circles to itself inducing \( g \). Let \( \mathcal{T} \) be a transition graph for \( \psi \). The cone \( \mathcal{H} \) is a cone whose cross section is the closure of the set of all \( t_n(\gamma) \), where \( \gamma \) is a loop in \( \mathcal{T} \).

As discussed above, this cross section is a \( \dim H_1(S, \mathbb{R}) \) dimensional convex polytope. As before, for each of its vertices we can find an extremal subgraph of \( \mathcal{T} \), and the proof now proceeds exactly as the proof of Theorem 2.1.

References


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