# FREE JÓNSSON-TARSKI ALGEBRAS

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Abstract. For any nonempty set $X$, the partition lattice $\mathrm{Eq}(X)$ can be embedded into the congruence lattice of the free Jónsson-Tarski algebra generated by $X$.

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Jónsson and Tarski [3] found a variety of algebras (here abbreviated as J-T algebras, a.k.a. Cantor algebras) with the property that every finitely generated free algebra is isomorphic to the 1-generated free algebra $F(1)$. We show that for any nonempty set $X$, the lattice of partitions (equivalence relations) on $X$ embeds into the congruence lattice $\mathrm{Con}\, F(X)$ of the free J-T algebra on $X$. This is one way of showing that a variety satisfies no congruence lattice identities; cf. [4].

A J-T algebra comes with a binary multiplication and 2 unary operations, satisfying

$$w = s \cdot t \text{ iff } s = \lambda(w) \text{ and } t = \rho(w)$$

whence they also satisfy

$$sa = tb \text{ implies } s = t \text{ and } a = b.$$

We will show that every element in a free J-T algebra has a unique, shortest (canonical) form as a term in $X$.

Let $X$ be a set of variables, and let $F(X)$ be the free J-T algebra generated by $X$. Toward showing that every element of $F(X)$ has a canonical form, we identify 3 sets of terms.

- $T$ is the set of all J-T terms on $X$,
- $G$ is the set of all $u \in T$ such that $u$ has no subterm of the form $\lambda(st)$ or $\rho(st)$,
- $R$ is the set of all $v \in G$ such that $v$ has no subterm of the form $\lambda s \cdot \rho s$.

Write $s \approx t$ to mean that $s$ and $t$ are terms that evaluate the same in $F(X)$. Then $s \rightsquigarrow s'$ means that $s \approx s'$ and $\ell(s') \leq \ell(s)$ and $s$ has been transformed to $s'$ by a valid application of the laws of J-T algebras.

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The transformation to canonical form will be in 2 steps. Starting with $s \in T$, the first step $s \overset{1}{\rightsquigarrow} s'$ finds $s' \in G$ with $s \approx s'$. The second step $s' \overset{2}{\rightsquigarrow} s''$ takes $s' \in G$ and reduces it to $s'' \in R$ with $s' \approx s''$. We will then show that $G$ with appropriate operations forms a J-T algebra.

**First reduction.** We want to reduce an arbitrary term $w$ to a term in $G$. Think of $G$ as the absolutely free groupoid on

$$Y = \{o_1 \dots o_k(x) : k \geq 0,\ o_i \in \{\lambda, \rho\},\ x \in X\}.$$

The reduction is done recursively from the outside in. Trivially $x \overset{1}{\rightsquigarrow} x$. Assume that for all shorter terms than $w$ we have $s \overset{1}{\rightsquigarrow} s'$ with $s' \in G$, that is, $s'$ can be written as a product (with parentheses) of elements from $Y$.

If $w \in G$, then $w \overset{1}{\rightsquigarrow} w$. So assume $w \notin G$.

If $w = st$, then $w \overset{1}{\rightsquigarrow} s't'$ where $s \overset{1}{\rightsquigarrow} s'$ and $t \overset{1}{\rightsquigarrow} t'$, and $s't' \in G$.

If $w = o_1 \dots o_{k-1}\lambda(st)$, then $w \approx o_1 \dots o_{k-1}s$. The RHS may not be in $G$, but it is shorter, and thus reduces to some $w' \in G$. Let $w \overset{1}{\rightsquigarrow} w'$.

If $w = o_1 \dots o_{k-1}\rho(st)$, then $w \approx o_1 \dots o_{k-1}t$, which again is shorter, and reduces to some $w' \in G$. Again $w \overset{1}{\rightsquigarrow} w'$.

**Second reduction.** Now we take a term $u \in G$ and reduce it to a term $u' \in R$, so that $u'$ has no subterm of the form $\lambda s \cdot \rho s$. This reduction is done recursively from the inside out. If $u \in Y$, then $u \in R$. Otherwise $u = st$ is a product, and we can assume that $s \overset{2}{\rightsquigarrow} s' \in R$ and $t \overset{2}{\rightsquigarrow} t' \in R$. Then $s't'$ is in $R$ unless there is some $v$ such that $s' = \lambda v$ and $t' = \rho v$. In the latter event, $s't' \approx v$, which as a subterm of $s'$ is in $R$. So let either $u \overset{2}{\rightsquigarrow} s't'$ or $u \overset{2}{\rightsquigarrow} v$, as appropriate.

Observe that the second reduction does not change the variables: if $u \overset{2}{\rightsquigarrow} u'$, then $\operatorname{var}(u') = \operatorname{var}(u)$.

Now we claim that operations can be defined on $R$ to make it a J-T algebra. Every term in $R$ is either in $Y$ or uniquely a product $st$ of shorter terms in $R$. Define operations $\cdot$, $\lambda$, $\rho$ on $R$ thusly: for $y$, $y' \in Y$ and $st$, $uv \in R$,

$$\begin{aligned} y \cdot y' &= \begin{cases} z & \text{if } y = \lambda z \text{ and } y' = \rho z, \\ yy' & \text{otherwise} \end{cases} \\ y \cdot st &= y(st) \\ st \cdot y &= (st)y \\ st \cdot uv &= (st)(uv). \end{aligned}$$

That is, the formal product is in $R$ except when both terms are in $Y$, in which case it need not be. Then define

$$\lambda(o_1 \dots o_k(x)) = (\lambda o_1 \dots o_k)(x)$$
$$\lambda(st) = s$$

and symmetrically

$$\rho(o_1 \dots o_k(x)) = (\rho o_1 \dots o_k)(x)$$
$$\rho(st) = t$$

Again, the result is in $R$.

The algebra $R = \langle R, \cdot, \lambda, \rho \rangle$ is a J-T algebra generated by $X$, and has been constructed using only the laws that hold in all J-T algebras. Thus $R \cong F(X)$.

In other words, every J-T term in the variables $X$ is equivalent to exactly one term in $R$. Moreover, if $u \in R$ and $u \approx v$, then we can apply the reduction $v \overset{1}{\rightsquigarrow} v' \overset{2}{\rightsquigarrow} v'' \in R$. Then $v'' = u$ and of course $\ell(v'') \leq \ell(v)$, so that $u$ is the unique shortest term representing $u/\approx$, that is, $u$ is the canonical form.

For a partition $\pi$ on $X$, let $\pi(\mathbf{x})$ be the map that identifies $\pi$-related variables. Then define the map $h : \pi \mapsto \hat{\pi}$ of partitions to congruences such that

$$s \,\hat{\pi}\, t \quad \text{iff} \quad s(\mathbf{x}/\pi) \approx t(\mathbf{x}/\pi)$$

in the free J-T algebra. For example, if $\pi = [1|23|4]$ then

$$s \,\hat{\pi}\, t \quad \text{iff} \quad s(x, y, y, z) \approx t(x, y, y, z)$$

We claim that the map $h$ embeds $\Pi(X)$ into Con $F(X)$. It is easy to check that

- $\hat{\pi}$ is a congruence (the kernel of the natural homomorphism $F(X) \to F(X/\pi)$),
- $h$ is 1-1 (since it is on $X$),
- $h$ is order-preserving,
- $h$ preserves joins.

It remains to verify that $h$ preserves meets, i.e., $\hat{\theta} \wedge \hat{\varphi} \leq \widehat{\theta \wedge \varphi}$.

Let $(u, v) \in \hat{\theta} \wedge \hat{\varphi}$. We can take the algebra $R$ as our model of $F(X)$, so that $u$, $v \in R$. Then $u(\mathbf{x}/\theta) \approx v(\mathbf{x}/\theta)$ and $u(\mathbf{x}/\varphi) \approx v(\mathbf{x}/\varphi)$, and we want to show that $u(\mathbf{x}/(\theta \wedge \varphi)) \approx v(\mathbf{x}/(\theta \wedge \varphi))$.

Case 1: $u$, $v \in Y$, say $u = o_1 \dots o_k(x_1)$ and $u = p_1 \dots p_m(x_2)$. From $u(\mathbf{x}/\theta) \approx v(\mathbf{x}/\theta)$ and canonical form, we see that $k = m$, $o_i = p_i$ for all $i$, and $x_1 \,\theta\, x_2$. Similarly from $u(\mathbf{x}/\varphi) \approx v(\mathbf{x}/\varphi)$ we get $x_1 \,\varphi\, x_2$, so that $x_1 \; \theta \wedge \varphi \; x_2$. Thus $u \; \widehat{\theta \wedge \varphi} \; v$.

Case 2: $u = s(\mathbf{x}) \cdot t(\mathbf{x})$ and $v \in Y$, say $v = o_1 \dots o_k(x_1)$. Since $u(\mathbf{x}) \in R$ we have $u(\mathbf{x}/\theta) \in G$. Also $u(\mathbf{x}/\theta) \approx v(\mathbf{x}/\theta)$ which, assuming we choose $x_1$ as the representative of $x_1/\theta$, implies $u(\mathbf{x}/\theta) \overset{2}{\rightsquigarrow} o_1 \dots o_k(x_1)$. By the observation above that $u \overset{2}{\rightsquigarrow} u'$ does not change $\mathrm{var}(u)$, $u$ can only depend on variables in $x_1/\theta$. Similarly, $\mathrm{var}(u) \subseteq x_1/\varphi$, so that $\mathrm{var}(u) \subseteq x_1/(\theta \wedge \varphi)$. Then the reduction $u(\mathbf{x}/(\theta \wedge \varphi)) \overset{2}{\rightsquigarrow} o_1 \dots o_k(x_1)$ holds, whence $u \; \widehat{\theta \wedge \varphi} \; v$.

Case 3: $u = sa$ and $v = tb$. Then we get

$$s(\mathbf{x}/\theta) \cdot a(\mathbf{x}/\theta) = t(\mathbf{x}/\theta) \cdot b(\mathbf{x}/\theta)$$
$$s(\mathbf{x}/\varphi) \cdot a(\mathbf{x}/\varphi) = t(\mathbf{x}/\varphi) \cdot b(\mathbf{x}/\varphi).$$

By the cancellation property for multiplication, this implies

$$s(\mathbf{x}/\theta) \approx t(\mathbf{x}/\theta) \text{ and } s(\mathbf{x}/\varphi) \approx t(\mathbf{x}/\varphi)$$
$$a(\mathbf{x}/\theta) \approx b(\mathbf{x}/\theta) \text{ and } a(\mathbf{x}/\varphi) \approx b(\mathbf{x}/\varphi).$$

These are shorter words in $\hat{\theta} \wedge \hat{\varphi}$, so we conclude by induction that $s(\mathbf{x}/(\theta \wedge \varphi)) \approx t(\mathbf{x}/(\theta \wedge \varphi))$ and $a(\mathbf{x}/(\theta \wedge \varphi)) \approx b(\mathbf{x}/(\theta \wedge \varphi))$. Hence $u(\mathbf{x}/(\theta \wedge \varphi)) \approx v(\mathbf{x}/(\theta \wedge \varphi))$, as was to be shown.

**Theorem 1.** *The map $h$ embeds $\Pi(X)$ into* Con $F(X)$.

Recent papers of Cardó [1, 2] explore groupoids (magmas) in which the operation is injective: $sa = tb$ implies $s = t$ and $a = b$. Reducts of J-T algebras of course have this property.

## References


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