PRIMITIVE LATTICE VARIETIES

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Abstract. A variety is primitive if every subquasivariety is equational, i.e., a subvariety. In this note, we explore the connection between primitive lattice varieties and Whitman's condition (W). For example, if every finite subdirectly irreducible lattice in a locally finite variety $\mathcal{V}$ satisfies Whitman's condition (W), then $\mathcal{V}$ is primitive. This allows us to construct infinitely many sequences of primitive lattice varieties, and to show that there are $2^{\aleph_0}$ such varieties. Some lattices that fail (W) also generate primitive varieties. But if I is a (W)-failure interval in a finite subdirectly irreducible lattice $\mathbf{L}$, and $\mathbf{L}[I]$ denotes the lattice with I doubled, then $\mathbb{V}(\mathbf{L}[I])$ is never primitive.

Keywords: lattice variety, subvariety, subquasivariety, Whitman's condition

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1. Introduction

A justifiably famous result of Grätzer and Lakser [10] is that the variety $\mathbb{V}(\mathbf{M}_{3,3})$ contains uncountably many subquasivarieties; indeed, it is Q-universal [1]. Our goal here is to show that there are many finite lattices $\mathbf{L}$ such that every subquasivariety of $\mathbb{V}(\mathbf{L})$ is a variety, in which case the lattice of subquasivarieties $\mathrm{L_q}(\mathbb{V}(\mathbf{L}))$, being the same as the lattice of subvarieties, is finite and distributive.

A quasivariety $\mathcal{Q}$ is said to be *primitive* if every subquasivariety $\mathcal{K} \leq \mathcal{Q}$ is equational relative to $\mathcal{Q}$. An algebra $\mathbf{A}$ is said to be *weakly projective* in a class $\mathcal{K}$ if whenever $\mathbf{A}$ is a homomorphic image of an algebra $\mathbf{B} \in \mathcal{K}$, then $\mathbf{A}$ embeds in $\mathbf{B}$. The characterization of primitive, locally finite quasivarieties is due to Gorbunov [8, 9] and Slavík [20], independently.

Theorem 1. *A locally finite quasivariety $\mathcal{Q}$ of finite type is primitive if and only if every finite $\mathcal{Q}$-subdirectly irreducible algebra in $\mathcal{Q}$ is weakly projective in $\mathcal{Q}_{\mathrm{fin}}$, the class of finite algebras in $\mathcal{Q}$. Moreover, if $\mathcal{Q}$ is primitive, then the lattice of subquasivarieties $\mathrm{L_q}(\mathcal{Q})$ is distributive.*

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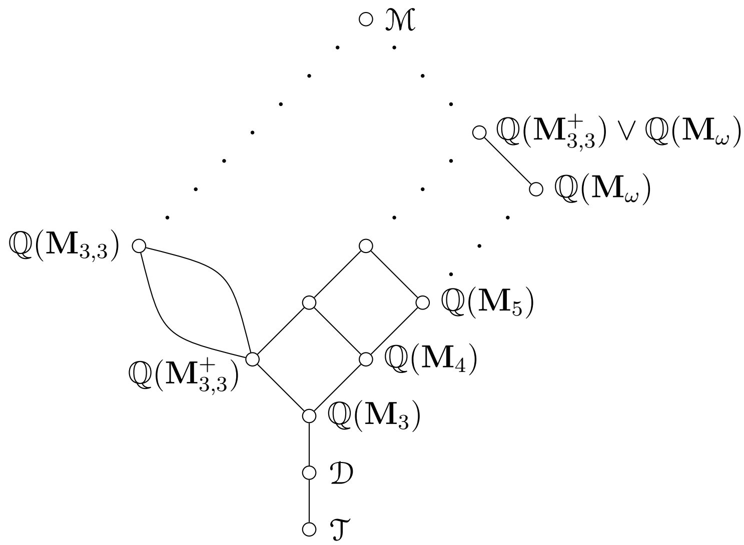


FIGURE 1. The lattice $\mathrm{L_q}(\mathcal{M})$ of quasivarieties of modular lattices. The interval $[\mathbb{Q}(\mathbf{M}_{3,3}^+), \mathbb{Q}(\mathbf{M}_{3,3})]$ contains uncountably many subquasivarieties, while the ideal $\downarrow(\mathbb{Q}(\mathbf{M}_{3,3}^+) \vee \mathbb{Q}(\mathbf{M}_\omega))$ is countable and distributive. The quasivarieties $\mathbb{Q}(\mathbf{M}_k)$ $(1 \leq k \leq \omega)$ and $\mathcal{D}$, $\mathcal{T}$ are in fact varieties. Likewise $\mathbb{Q}(\mathbf{M}_{3,3})$ is a variety, but nothing else in the interval $[\mathbb{Q}(\mathbf{M}_{3,3}^+), \mathbb{Q}(\mathbf{M}_{3,3})]$ is one.

In this note, we concentrate on the case when $\mathcal{Q}$ is a lattice variety. In that situation, *equational relative to* $\mathcal{Q}$ becomes just *equational*, and $\mathcal{Q}$-*subdirectly irreducible* is just *subdirectly irreducible*. We are then asking when every subquasivariety of a variety is, in fact, a subvariety.

For lattices, we have two wonderful tools at our disposal. The first is Jónsson's Lemma: *If* $\mathbf{L}$ *is a finite lattice, then the subdirectly irreducible lattices in* $\mathbb{V}(\mathbf{L})$ *are contained in* $\mathbb{HS}(\mathbf{L})$. Note that for lattices, or more generally any class of algebras with idempotent elements, $\mathbb{V}(\{\mathbf{L}_1, \ldots, \mathbf{L}_m\}) = \mathbb{V}(\mathbf{L}_1 \times \cdots \times \mathbf{L}_m)$, and likewise for quasivarieties $\mathbb{Q}(\{\mathbf{L}_1, \ldots, \mathbf{L}_m\}) = \mathbb{Q}(\mathbf{L}_1 \times \cdots \times \mathbf{L}_m)$. Thus every finitely generated lattice variety or quasivariety is generated by a single lattice.

Recall Whitman's condition from the solution of the word problem for free lattices [21]:

$$(W) \quad \text{if } s = \bigwedge_{i=1}^{m} s_i \leq \bigvee_{j=1}^{n} t_j = t, \text{ then either } s_i \leq t \text{ for some } i, \text{ or } s \leq t_j \text{ for some } j.$$

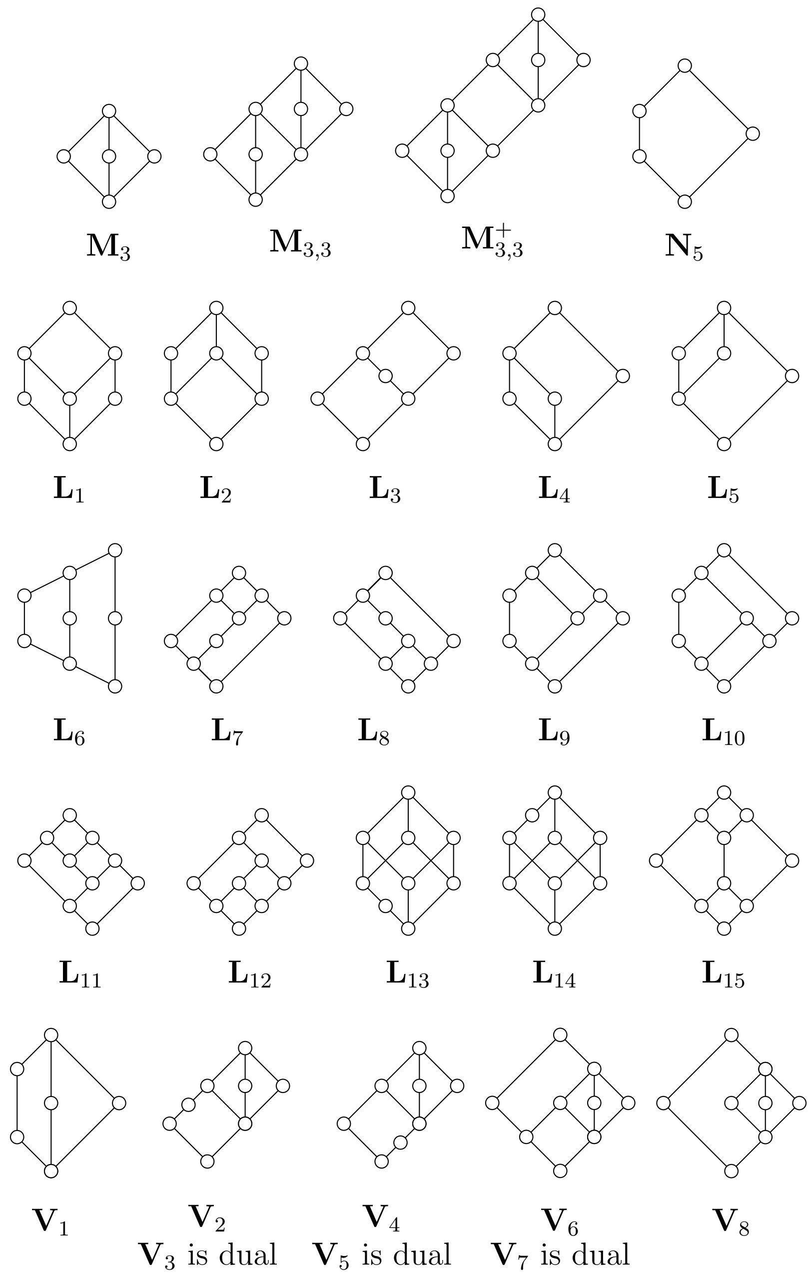


FIGURE 2. The lattices $\mathbf{M}_3, \mathbf{M}_{3,3}, \mathbf{M}_{3,3}^+, \mathbf{N}_5$, and the lattices $\mathbf{L}_1, \dots, \mathbf{L}_{15}$ and $\mathbf{V}_1, \dots, \mathbf{V}_8$ that generate join irreducible covers of $\mathbb{V}(\mathbf{N}_5)$ and $\mathbb{V}(\mathbf{M}_3, \mathbf{N}_5)$, respectively.

Jónsson, McKenzie, and Kostinsky linked (W) together with bounded homomorphisms to characterize finitely generated projective lattices; see Chapter II of [6] for the arguments and discussion. Davey and Sands [3] added the observation that for lattices with no infinite chains, all homomorphisms are bounded, to yield: *a finite lattice is projective in the class of finite lattices if and only if it satisfies* (W). Combining these conditions gives a sufficient criterion for the variety generated by a finite lattice to be a primitive quasivariety.

Theorem 2. *Assume that* $\mathbf{L}$ *is a finite lattice with the property that every subdirectly irreducible lattice in* $\mathbb{HS}(\mathbf{L})$ *satisfies* (W). *Then the variety* $\mathbb{V}(\mathbf{L})$ *is a primitive quasivariety, i.e., every subquasivariety is equational.*

For modular lattices, Theorem 2 applies only to the lattices $\mathbf{M}_k$ (including by extension $\mathbf{M}_\omega$; see Theorem 20). Note that $\mathbf{M}_{3,3}$ fails (W). But many nonmodular lattices generate primitive varieties, for example, the pentagon $\mathbf{N}_5$ and all 16 lattices that generate a cover of $\mathbb{V}(\mathbf{N}_5)$ in the lattice of lattice varieties.

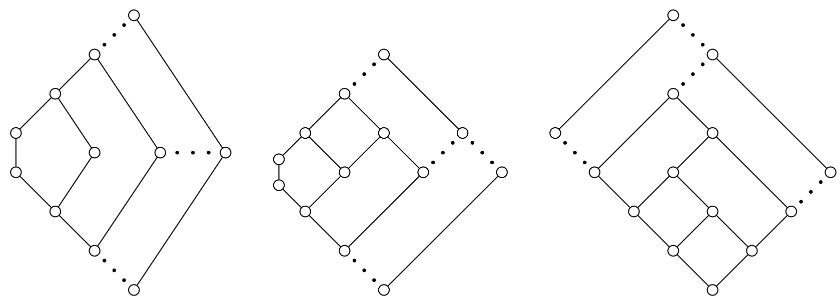
A locally finite lattice is *inherently Whitman* (IW) if every subdirectly irreducible (s.i.) lattice in $\mathbb{HS}(\mathbf{L})$ satisfies (W). We say that a variety $\mathcal{V}$ is *inherently Whitman* if it is locally finite and every finite s.i. lattice in $\mathcal{V}$ is inherently Whitman. Theorem 2 states that an inherently Whitman variety is primitive.

Later, we will require a stronger version: a finite lattice is *strongly inherently Whitman* (SIW) if *every* lattice in $\mathbb{HS}(\mathbf{L})$ satisfies (W). Note that SIW implies IW.

Looking closer at the bottom of the lattice Λ of lattice varieties we find, besides the varieties $\mathbb{V}(\mathbf{M}_k)$, five more chains of primitive lattice varieties: $\mathbb{V}(\mathbf{L}_j^k)$ for $j \in \{6, 9, 10, 11, 12\}$ and $k \geq 1$. Figure 3 illustrates $\mathbf{L}_6^k$, $\mathbf{L}_9^k$, and $\mathbf{L}_{12}^k$; $\mathbf{L}_{10}^k$ is dual to $\mathbf{L}_9^k$, and $\mathbf{L}_{11}^k$ is dual to $\mathbf{L}_{12}^k$. These lattices are all inherently Whitman. Here we use the standard numbering scheme from [14, 16].

However, most of the remaining lattices generating covers of covers of $\mathbb{V}(\mathbf{N}_5)$ fail (W). This in itself does not prohibit a lattice from generating a primitive variety, and in Section 5 we will see that some of them do so.

The bottom of the lattice $\mathrm{L_q}(\mathcal{L})$ of lattice varieties is very thin. There is one atom (distributive lattices) and 2 varieties of height 2 ($\mathbb{V}(\mathbf{M}_3)$ and $\mathbb{V}(\mathbf{N}_5)$), 18 varieties of height 3, etc., until eventually the number of elements of a given height becomes infinite [19]. For a recent summary, see Jipsen and Rose [14].



$\mathbf{L}_6^k$ $\mathbf{L}_9^k$ $\mathbf{L}_{12}^k$

FIGURE 3. Some chains of primitive lattice varieties.

The lattice $\mathrm{L_q}(\mathbb{V}(\mathbf{L}))$ for a primitive lattice variety inherits these restrictions, so not every finite distributive lattice is obtained as a lattice of subquasivarieties of a primitive lattice variety. If $\mathbb{V}(\mathbf{L})$ is a primitive lattice variety, then it has one atom, at most 2 varieties of height 2, at most 17 varieties of height 3 (not including $\mathbb{V}(\mathbf{M}_{3,3})$), etc.

The terminology in the literature with regard to primitive quasivarieties varies considerably. Bergman [2] defines a quasivariety $\mathcal{Q}$ to be *structurally complete* if every proper subquasivariety generates a proper subvariety of $\mathbb{V}(\mathcal{Q})$. The expression originates from logic, where a propositional logic is called structurally complete if every proper extension by a logical rule adds new tautologies to the logic. Algebraic logic sets up a correspondence between so-called algebraizable propositional logics and quasivarieties of algebras, with basic operations given by the logical connectives. A logic is *inherently structurally complete* if every extension is structurally complete. On the algebraic side, this notion corresponds exactly to the quasivariety being primitive, though some authors use the logical terminology, and Bergman [2] calls such quasivarieties *deductive.*

To avoid confusion, we also mention that a *primitive lattice* $\mathbf{L}$ as defined in [6, 12] does not imply that *the variety generated by* $\mathbf{L}$ *is primitive*, nor does the converse hold.

Section 2 refines the characterization of primitive lattice varieties, and explores the connection with Whitman's condition (W). This is exploited in Section 3, where we look at properties of inherently Whitman varieties and some constructions that preserve being IW. This allows us in Section 4 to construct infinitely many sequences of primitive lattice varieties like those in Figure 3. In Section 5, we investigate finite lattices $\mathbf{L}$ that fail (W) but still generate a primitive lattice variety. The methods developed there allow us to determine, for each s.i. lattice $\mathbf{L}$ with $|L| \leq 9$, whether $\mathbb{V}(\mathbf{L})$ is primitive; this is done in Section 6. Finally, Section 7 gives a few more lattice gluing constructions that preserve the property of generating a primitive variety.

2. Primitive lattice varieties and Whitman's condition

One can refine Theorem 1 for lattice varieties.

Theorem 3. *A locally finite lattice variety $\mathcal{V}$ is not primitive iff there exist finite $\mathbf{L}$, $\mathbf{L}' \in \mathcal{V}$ such that*

(1) *$\mathbf{L}$ is s.i.,*
(2) *$\mathbf{L} \not\leq \mathbf{L}'$, i.e., $\mathbf{L}'$ does not contain $\mathbf{L}$ as a sublattice,*
(3) *there is a surjective homomorphism $f : \mathbf{L}' \twoheadrightarrow \mathbf{L}$.*

If $\mathcal{V}$ is not primitive, we may also assume that

(4) *there is a (W)-failure $s \leq t$ in $\mathbf{L}$,*

and that there is a transversal $g : \mathbf{L} \to \mathbf{L}'$ of f such that

(5) *$T = g(L)$ generates $\mathbf{L}'$,*
(6) *$g(s) \not\leq g(t)$.*

Property (6) says that the given (W)-failure is corrected in $\mathbf{L}'$.

Proof. First assume that $\mathcal{V}$ contains $\mathbf{L}$, $\mathbf{L}'$ satisfying (1), (2), (3). If $\mathbf{L}$ is $\mathcal{K}$-s.i. for a quasivariety $\mathcal{K}$, then the class $\mathcal{X}_{\mathbf{L}}$ of lattices $\mathbf{T}$ in $\mathcal{K}$ such that $\mathbf{L} \not\leq \mathbf{T}$ is a subquasivariety. Indeed, if (s, t) is a critical pair, it is all lattices satisfying

$$(\text{diagram of } \mathbf{L}) \to s \approx t$$

where the *diagram* of $\mathbf{L}$ is a conjunction of equations encoding the join and meet tables of $\mathbf{L}$; see e.g. [11].

In our case, $\mathbf{L}$ is s.i. by (1), and $\mathbf{L}' \in \mathcal{X}_{\mathbf{L}}$ by (2) and the assumption $\mathbf{L}' \in \mathcal{V}$. However, $\mathcal{X}_{\mathbf{L}}$ is not closed under homomorphic images by (3). Thus $\mathcal{V}$ is not primitive, because it contains the subquasivariety $\mathcal{X}_{\mathbf{L}}$ which is not a variety.

Conversely, assume $\mathcal{V}$ is not primitive. Then there is a subquasivariety $\mathcal{Q} \leq \mathcal{V}$ that is a not a subvariety, i.e., $\mathbb{H}(\mathcal{Q}) \not\subseteq \mathcal{Q}$. There is a s.i. lattice $\mathbf{L} \in \mathbb{H}(\mathcal{Q}) \setminus \mathcal{Q}$. Then $\mathbf{L} \in \mathbb{H}(\mathcal{Q})$ means that (3) holds, while $\mathbf{L} \notin \mathcal{Q}$ implies (2).

Thus far we have only reproduced the proof of Theorem 1 for lattice varieties. Now we want to connect this to (W)-failures. For that purpose, we recall the standard sequence of maps used to find a retract $g : \mathbf{L} \to \mathbf{K}$ of a bounded homomorphism $f : \mathbf{K} \to \mathbf{L}$ when $\mathbf{L}$ satisfies (W). These ideas originated with McKenzie [17] and Jónsson [15]; we follow Section V.1 of [6], simplified for finite lattices and surjective homomorphisms.

So assume that $\mathbf{K}$ and $\mathbf{L}$ are finite (but not yet that $\mathbf{L}$ satisfies (W)), and let $f : \mathbf{K} \twoheadrightarrow \mathbf{L}$ be surjective. We seek an embedding $g : \mathbf{L} \to \mathbf{K}$ such that $gf = \mathrm{id}_{\mathbf{L}}$. Since homomorphisms between finite lattices are bounded, we can obtain a meet-preserving transversal $g_0 : \mathbf{L} \to \mathbf{K}$. For example, $g_0(a)$ could be the largest pre-image of a, for all $a \in L$. Then recursively define

$$g_{k+1}(x) = g_0(x) \wedge \bigwedge_{\bigvee U \geq x} \bigvee_{u \in U} g_k(u).$$

For the second term on the right side, we need only take the minimal (in the sense of refinement) nontrivial join covers $\bigvee U \geq x$, as other terms will be absorbed by these.

Lemma 4. *The sequence of maps g_k has these properties.*

(i) *$g_k(x) \geq g_{k+1}(x)$ for all $k \in \omega$, $x \in L$.*
(ii) *g_k is order-preserving.*
(iii) *$g_{k+1}(\bigvee V) \leq \bigvee_{v \in V} g_k(v)$ for any $V \subseteq L$.*
(iv) *If $T \subseteq L$ is a subset such that there is no (W)-failure $\bigwedge T \leq \bigvee U$ in $\mathbf{L}$, then $g_{k+1}(\bigwedge T) = \bigwedge_{t \in T} g_{k+1}(t)$.*

Proof. The proof of (i) and (ii) is by induction, while (iii) is clear.

For (iv), we copy the relevant part of the proof of Theorem 5.7 of [6]. Let $\bigwedge T = p$. We want to show that $g_{k+1}(p) \geq \bigwedge_{t \in T} g_{k+1}(t)$, the other direction holding by (ii). Now $g_{k+1}(p) = g_0(p) \wedge \bigwedge_{\bigvee U \geq p} \bigvee_{u \in U} g_k(u)$. We are given that g_0 preserves meets, and it suffices to consider nontrivial (even minimal nontrivial) join covers U of p. But if $p = \bigwedge T \leq \bigvee U$ nontrivially, then we can apply (W) at p to obtain $t_0 \leq \bigvee U$ for some $t_0 \in T$. That implies $g_{k+1}(t_0) \leq \bigvee_{u \in U} g_k(u)$. This argument applies to every meetand in the expression for $g_{k+1}(p)$, yielding the conclusion. □

Choose n such that $g_{n+1}(x) = g_n(x)$ for every $x \in L$, and let $g(x) = g_n(x)$. Then by (iii), g is a join-preserving transversal of f. When $\mathbf{L}$ satisfies (W), then by (iv), g also preserves meets. In that case, g is a retract and $\{g(x) : x \in L\}$ is a sublattice of $\mathbf{K}$ isomorphic to $\mathbf{L}$.

Now we return to the proof of Theorem 3. Assuming (1)–(3), form the standard sequence of maps $g_k : \mathbf{L} \to \mathbf{L}'$. Using the fact that f is a bounded homomorphism, we start with $g_0(x)$ being the largest pre-image of x, so that g_0 preserves meets. Then recursively define

$$g_{k+1}(x) = g_0(x) \wedge \bigwedge_{\bigvee U \geq x} \bigvee_{u \in U} g_k(u).$$

In view of (2), this sequence will fail to form a retraction. On the other hand, it always gives a transversal, and for large enough n, g_n

will preserve joins. Hence for some k, g_k will preserve meets while g_{k+1} will cease to do so. To simplify notation, assume $k = 0$, so that g_1 does not preserve meets.

Since g_1 preserves order but not meets, there are elements a, $b \in L$ such that $g_1(a) \wedge g_1(b) \not\leq g_1(a \wedge b)$. Since the left side is below $g_0(a) \wedge g_0(b) = g_0(a \wedge b)$, that means there exists $W_0 \subseteq L$ such that $a \wedge b \leq \bigvee W_0$ and

$$(\varkappa) \qquad g_0(a \wedge b) \wedge (\bigwedge_{\bigvee U \geq a} \bigvee_{u \in U} g_0(u)) \wedge (\bigwedge_{\bigvee V \geq b} \bigvee_{v \in V} g_0(v)) \not\leq \bigvee_{w \in W_0} g_0(w)$$

We claim that $a \wedge b \leq \bigvee W_0$ is a (W)-failure in $\mathbf{L}$ (to be shown below), which is (4). There are various ways to form the transversal, but the simplest is to let

$$g(x) = \begin{cases} g_0(x) & \text{if } x \leq \bigvee W_0 \\ g_1(x) & \text{otherwise.} \end{cases}$$

Then $(\varkappa)$ becomes (6). We may assume that $g(\mathbf{L})$ generates $\mathbf{L}'$, replacing it with a sublattice if necessary; this is (5).

To see that $a \wedge b \leq \bigvee W_0$ is a (W)-failure, first suppose $a \leq \bigvee W_0$. Then the right hand side becomes one of the meetands of $g_1(a)$, whence $g_1(a) \wedge g_1(b) \leq g_1(a) \leq \bigvee_{w \in W_0} g_0(w)$, a contradiction. Likewise if $b \leq \bigvee W_0$. On the other hand, if $a \wedge b \leq w_0$ for some $w_0 \in W_0$, then

$$g_1(a) \wedge g_1(b) \leq g_0(a) \wedge g_0(b) = g_0(a \wedge b) \leq g_0(w_0) \leq \bigvee_{w \in W_0} g_0(w)$$

again contradicting $(\varkappa)$. □

From just the first part of Theorem 3, or directly from Jónsson's Lemma, we have the following consequence.

Corollary 5. *Suppose the finite lattice* $\mathbf{K}$ *satisfies* (W) *and there is a s.i.* $\mathbf{L}$ *in* $\mathbb{HS}(\mathbf{K})$ *that fails* (W). *Then* $\mathbb{V}(\mathbf{K})$ *is not primitive.*

The lattice $\mathbf{H}[m]$ in Figure 4 is the only example of this corollary with 9 elements.

The theorem suggests two questions. Let $\mathbf{L}[I]$ denote the Day doubling construction [4, 5] applied to an interval I of $\mathbf{L}$. Doubling a single element m is denoted $\mathbf{L}[m]$.

(A) Can $\mathbf{L}'$ be anything besides $\mathbf{L}[I]$?
(B) Is it possible to have $\mathbf{L} \leq \mathbf{L}[I]$ for a (W)-failure interval?

For (A), the answer is YES. Figure 5 gives a situation where we might not want $\mathbf{L}' = \mathbf{L}[I]$, though we need better examples. Figure 6 illustrates how one finds $\mathbf{L}'$ in general. Given the setup of Theorem 3,

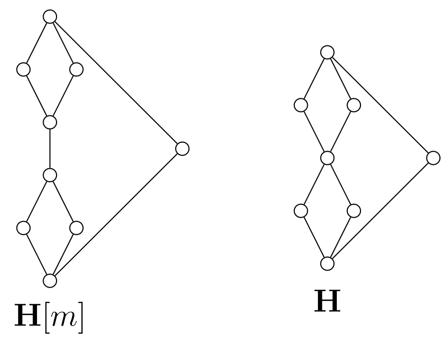


FIGURE 4. The lattice $\mathbf{H}[m]$ is subdirectly irreducible and satisfies (W), while its image $\mathbf{H} = \mathbf{H}[m]/\mu$ is s.i. and fails (W). By Corollary 5, $\mathbb{V}(\mathbf{H}[m])$ is not primitive, but in Section 5 we will see that $\mathbb{V}(\mathbf{H})$ is primitive.

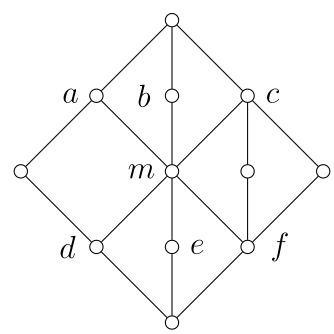


FIGURE 5. Suppose for some reason we ignore the $\mathbf{M}_{3,3}$'s and want to fix the (W)-failure $m = a \wedge b = e \vee f$ in the variety of modular lattices. Doubling m generates pentagons, so we must do something different. Doubling the larger interval $[d, c]$ fixes this (W)-failure in modular lattices (even $\mathbb{V}(\mathbf{L})$), allowing us to apply Theorem 3.

with $s \leq t$ a (W)-failure in the s.i. lattice $\mathbf{L} \in \mathcal{V}$, let T be the transversal of item (5). Note T could be regarded as a set, or an ordered set, or a partial lattice. Form the free lattice (or finitely presented lattice) $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(T)$. In the congruence lattice $\mathrm{Con}\, \mathbf{F}_{\mathcal{V}}(T)$, locate the kernel λ of the map onto $\mathbf{L}$, and the least congruence γ such that $(s \wedge t, t) \in \gamma$, that is, $s \leq t$ modulo γ. Thus $\gamma \leq \lambda$. Then consider congruences λ' such that $\lambda' < \lambda$ and $\gamma \nleq \lambda'$. If for some such λ' it happens that $\mathbf{L}$ does not embed into $\mathbf{F}/\lambda'$, then $\mathbf{F}/\lambda'$ witnesses that $\mathcal{V}$ is not a primitive variety. If, on the other hand, the above process fails for all choices of $\mathbf{L}$ s.i. in $\mathcal{V}$ and $s \leq t$ a (W)-failure in $\mathbf{L}$, then $\mathcal{V}$ is primitive.

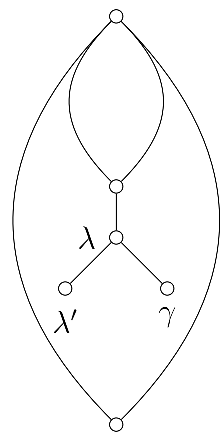


FIGURE 6. How to find $\mathbf{L}'$ as $\mathbf{F}_{\mathcal{V}}(T)/\lambda'$.

For (B), the answer is NO according to the following theorem, which Alan Day may have known.

Theorem 6. *If* $\mathbf{L}$ *is a finite lattice and* I *a* (W)*-failure interval in* $\mathbf{L}$*, then* $\mathbf{L} \not\leq \mathbf{L}[I]$.

Proof. Suppose $\varepsilon : \mathbf{L} \leq \mathbf{L}[I]$. There is also the map $f : \mathbf{L}[I] \to \mathbf{L}$ collapsing the doubled interval.

First consider when the (W)-failure in $\mathbf{L}$ is a doubly reducible element m. Then ε must map doubly reducible elements in $\mathbf{L}$ one-to-one to doubly reducible elements in $\mathbf{L}[m]$. But if $\mathbf{L}$ has say k doubly reducible elements, then $\mathbf{L}[m]$ has $k-1$, so that is impossible.

Now assume that $I = [s,t]$ is an interval of length $\ell(I) = p > 0$, where $\ell(I)$ is the length of longest chain in I. Let $F_{\geq p}(\mathbf{L})$ be the set of all (W)-failure intervals $[u,v]$ in $\mathbf{L}$ of length $\geq p$, and define $F_{\geq p}(\mathbf{L}[I])$ correspondingly. Observe that if $J = [a,b]$ is in $F_{\geq p}(\mathbf{L})$, then $\varepsilon J \subseteq [\varepsilon a, \varepsilon b]$ is in $F_{\geq p}(\mathbf{L}[I])$.

Lemma 7. *If* $K \in F_{\geq p}(\mathbf{L}[I])$*, then* $f(K) \not\subseteq I$.

Proof. Suppose $K = [u,v] \in F_{\geq p}(\mathbf{L}[I])$ and $f(K) \subseteq I$. We can write u as a meet of meet irreducibles, and v as a join of join irreducibles. The new join irreducible element in $\mathbf{L}[I]$ is $(s,1)$, while the new meet irreducible element is $(t,0)$. So for some $P \subseteq \mathrm{M}(\mathbf{L})$ and $Q \subseteq \mathrm{J}(\mathbf{L})$, one of 4 things happens:

- $\bigwedge P = u \leq v = \bigvee Q$,
- $(t,0) \wedge \bigwedge P = u \leq v = \bigvee Q$,

- $\bigwedge P = u \leq v = \bigvee Q \vee (s, 0)$,
- $(t, 0) \wedge \bigwedge P = u \leq v = \bigvee Q \vee (s, 1)$.

Each case leads to a contradiction.

In the first case, $u = (\bigwedge P, 1)$ and $v = (\bigvee Q, 0)$, so $u \nleq v$.

In the second case, $u = (t \wedge \bigwedge P, 0)$ and $v = (\bigvee Q, 0)$. Since $[u, v]$ is a (W)-failure, we must have $\bigvee Q < t$. That implies $\ell(K) < p$.

The third case is dual to the second.

In the fourth case, $u = (t \wedge \bigwedge P, 0)$ and $v = (s \vee \bigvee Q, 1)$. To have a (W)-failure, we must have $t \wedge \bigwedge P > s$ and $s \vee \bigvee Q < t$. Thus $\ell(K) \leq \ell(I) - 2 + 1 = p - 1 < p$. □

Now we have $\varepsilon : F_{\geq p}(\mathbf{L}) \to F_{\geq p}(\mathbf{L}[I])$ one-to-one, and $f : F_{\geq p}(\mathbf{L}[I]) \to F_{\geq p}(\mathbf{L})$ one-to-one. But I is in $F_{\geq p}(\mathbf{L})$ and not in the range of $f\varepsilon$, so that is impossible. □

When $\mathbf{L}$ is s.i. and I is a (W)-failure interval, we can have either $\mathbf{L}[I] \in \mathbb{V}(\mathbf{L})$ as with $\mathbf{M}_{3,3}$, or $\mathbf{L}[I] \notin \mathbb{V}(\mathbf{L})$ as with the lattice $\mathbf{H}[m]$ of Figure 4. Theorems 3 and 6 combine to show that in the former case, $\mathbb{V}(\mathbf{L})$ is not primitive.

Corollary 8. *If $\mathbf{L}$ is s.i., I is a (W)-failure interval in $\mathbf{L}$, and $\mathbf{L}[I] \in \mathbb{V}(\mathbf{L})$, then $\mathbb{V}(\mathbf{L})$ is not primitive.*

Besides $\mathbf{M}_{3,3}$, this corollary applies to the lattice $\mathbf{K}_6$ of Figure 9. One can make many similar examples by gluing 2 lattices so that the glued sum $\mathbf{L}$ is s.i. and the gluing interval I is a (W)-failure. See Lemma 24.

In this section, we have dealt with lattices $\mathbf{L}$ that are *weakly projective* in a locally finite variety $\mathcal{V}$, i.e., $\mathbf{L} \in \mathcal{V}$ and if $\mathbf{L}$ is a homomorphic image of a lattice $\mathbf{K} \in \mathcal{V}$, then $\mathbf{L} \leq \mathbf{K}$. Finite lattices that satisfy (W) are in fact *projective* in a locally finite variety, that is, retracts of a relatively free lattice. It is easy to find examples showing that these notions are distinct for general algebras. For example, in the variety $\mathcal{W}$ of 1-unary algebras satisfying $f^3x \approx f^2x$, the 2-element algebra with $a \neq f(a) = f^2(a)$ is weakly projective, but not projective. As we will see, lattices that fail (W) can be projective in some locally finite varieties, and not in others. But we know of no lattice that is weakly projective in a locally finite variety, without being projective in that variety. This is question (1) in Section 8.

Keith Kearnes (private communication) provided a partial answer with the following nice argument.

Theorem 9. *Let $\mathbf{K}$ be a finite s.i. algebra in a congruence distributive variety. If $\mathbf{K}$ is weakly projective in $\mathbb{V}(\mathbf{K})$, then $\mathbf{K}$ is projective in $\mathbb{V}(\mathbf{K})$.*

Proof. Let $\mathcal{V} = \mathbb{V}(\mathbf{K})$ and let $\mathbf{F}$ be a finitely generated $\mathcal{V}$-free algebra that has a homomorphism onto $\mathbf{K}$. By weak projectivity, $\mathbf{K}$ is embeddable in $\mathbf{F}$. Let $\theta \in \operatorname{Con} \mathbf{F}$ be maximal on $\mathbf{F}$ for the property $\theta|_{\mathbf{K}} = 0$. This implies that $\mathbf{F}/\theta$ is an essential extension of $\mathbf{K}$. In particular, $|\mathbf{K}| \le |\mathbf{F}/\theta|$.

Since $\mathbf{K}$ is s.i., and $\mathbf{F}/\theta$ is an essential extension of $\mathbf{K}$, $\mathbf{F}/\theta$ is also s.i. By Jónsson's Lemma, $\mathbf{F}/\theta \in \mathbb{HS}(\mathbf{K})$, so $|\mathbf{F}/\theta| \le |\mathbf{K}|$. Altogether this means that $|\mathbf{K}| = |\mathbf{F}/\theta|$. Since $\mathbf{K}$ is embeddable in $\mathbf{F}/\theta$ it follows that $\mathbf{K} \cong \mathbf{F}/\theta$. In fact, this means that the embedding of $\mathbf{K}$ into $\mathbf{F}$ is a section of the natural map $\mathbf{F} \to \mathbf{F}/\theta$. This forces $\mathbf{K}$ to be a retract of $\mathbf{F}$, so $\mathbf{K}$ is projective in $\mathcal{V}$. □

3. Inherently Whitman varieties

In this section, we will see how to construct inherently Whitman varieties. Theorem 2 and Corollary 5 combine to yield the following.

Lemma 10. *Let* $\mathbf{L}$ *be a finite lattice that satisfies* (W)*. Then* $\mathbb{V}(\mathbf{L})$ *is primitive iff* $\mathbf{L}$ *is IW.*

The next two results at least show that IW varieties are not rare.

Theorem 11. *Finitely generated, inherently Whitman varieties form an ideal in the lattice* Λ *of lattice varieties.*

Proof. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are such, so that every s.i. in each satisfies (W). By Jónsson's Lemma, the s.i.'s in $\mathcal{P} \vee \mathcal{Q}$ are in one or the other. □

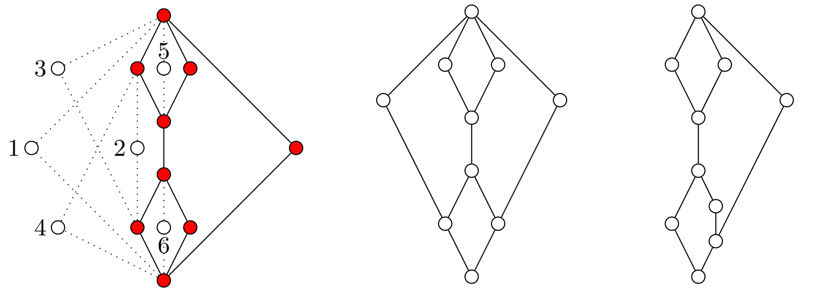
However, the join of primitive lattice varieties need not be primitive, so they do not form an ideal in Λ; see Corollary 28.

All s.i. lattices of cardinality ≤ 12 have been computed [13] and this data is useful for finding small IW lattices (see Table 1 in Section 6).

Theorem 12. *If* $\mathbf{L}$ *is s.i. and satisfies* (W) *and* $|L| \le 8$*, then* $\mathbf{L}$ *is inherently Whitman.*

So you could be forgiven for conjecturing: *If* $\mathbf{L}$ *is a finite subdirectly irreducible lattice satisfying* (W)*, then* $\mathbf{L}$ *is inherently Whitman.* Pardoned, but not exonerated: Figure 4 gives a 9-element s.i. lattice $\mathbf{H}[m]$ satisfying (W), but $\mathbf{H} = \mathbf{H}[m]/\mu$ is still s.i. and does not satisfy (W). This is the unique smallest such example, but there are 10 more (similar) examples with $|L| = 10$ (see Figure 7).

Let us consider some constructions inspired by Ježek and Slavík [12]. (See [6], pp. 129–132.) There will be more constructions in Section 7. For $\mathbf{L}$ a finite lattice and an element $a \in L$:



$\mathbf{J}_1$–$\mathbf{J}_6$ $\mathbf{J}_7 = \mathbf{J}_8^d$ $\mathbf{J}_9 = \mathbf{J}_{10}^d$

FIGURE 7. All ten s.i. lattices $\mathbf{J}_1$–$\mathbf{J}_{10}$ with 10 elements that satisfy (W) but have a s.i. homomorphic image that fails (W). For $\mathbf{J}_i$ $(i = 1, \ldots, 6)$ add element i connected by dotted lines. $\mathbf{L}^d$ denotes the dual lattice of $\mathbf{L}$.

- $C(\mathbf{L})$ is $L \cup \{c\}$ with $0_L < c < 1_L$ and c incomparable to x for the remaining elements of L, so that c a complement of every $x \in L \setminus \{0_L, 1_L\}$.
- $N(\mathbf{L})$ is the parallel sum of $\mathbf{L}$ and a 1-element lattice, i.e., $N(\mathbf{L}) = L \cup \{i, c, z\}$ with i the new top, z the new bottom, c a complement of everything in L.
- $U(\mathbf{L}, a)$ is the *wing-up* at a, i.e., $U(\mathbf{L}) = L \cup \{i, c\}$ with i the new top, $a < c < i$.
- Dually, $D(\mathbf{L}, a)$ is the *wing-down* at a, i.e., $D(\mathbf{L}) = L \cup \{d, z\}$ with z the new bottom, $a > d > z$.
- $Q(\mathbf{L}, a) = D(U(\mathbf{L}, a), c)$.
- Dually, $Q^*(\mathbf{L}, a) = U(D(\mathbf{L}, a), d)$.

These constructions are sketched schematically in Figure 8.

Let us consider when the constructions preserve subdirect irreducibility and Whitman's condition.

Theorem 13. *Let* $\mathbf{L}$ *be a finite lattice.*

(1) *If* $|L| > 2$ *and* $\mathbf{L}$ *is s.i., then* $C(\mathbf{L})$ *is s.i.*
(2) $C(\mathbf{L})$ *satisfies* (W) *iff* $\mathbf{L}$ *satisfies* (W).

Moreover, $C(\mathbf{L})$ *is strongly inherently Whitman iff* $\mathbf{L}$ *is strongly inherently Whitman.*

For example, $C(\mathbf{M}_k) = \mathbf{M}_{k+1}$. But you can start with any strongly inherently Whitman s.i. lattice except $\mathbf{2}$.

Proof. Claims (1) and (2) are clear. In (1), remember than every s.i. lattice except $\mathbf{2}$ has at least 5 elements. Note that the converse of (1) is

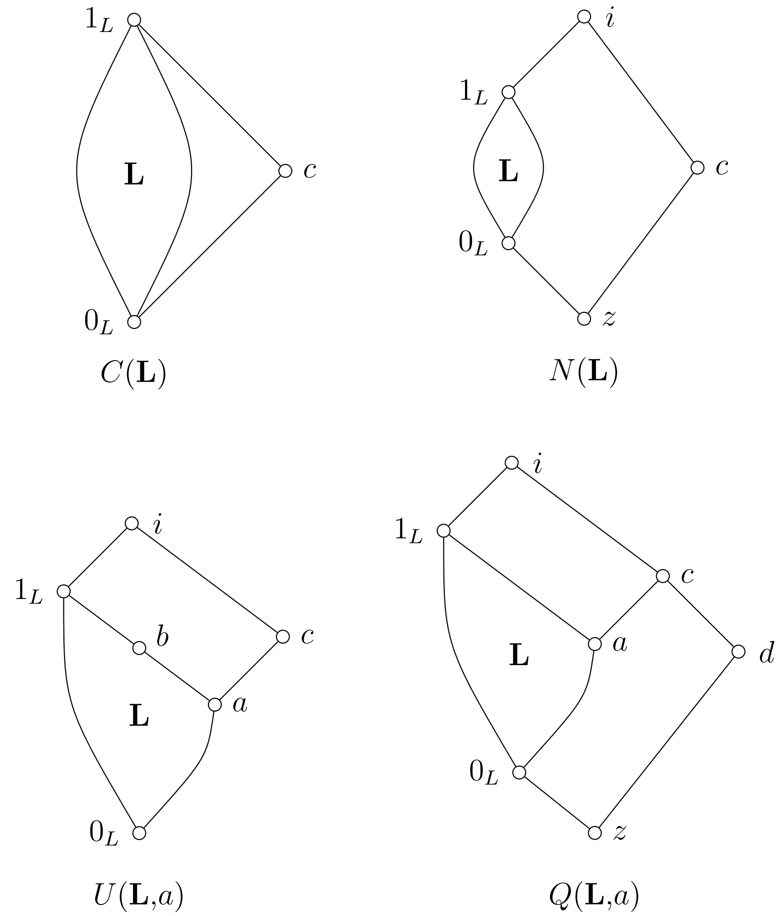


FIGURE 8. The constructions $C(\mathbf{L})$, $N(\mathbf{L})$, $U(\mathbf{L}, a)$, and $Q(\mathbf{L}, a)$

false, the lattices $\mathbf{H}$ and $\mathbf{H}[m]$ of Figure 4 being examples where $C(\mathbf{L})$ is s.i. for $\mathbf{L}$ subdirectly reducible.

Assume $\mathbf{L}$ is SIW. Consider a sublattice $\mathbf{T} \leq C(\mathbf{L})$, and a factor lattice $\mathbf{R} = \mathbf{T}/\theta$. If $c \notin T$, then $\mathbf{T}/\theta \in \mathbb{HS}(\mathbf{L})$, whence it satisfies (W). So assume $c \in T$. If $c\,\theta\,0$ or $c\,\theta\,1$, then $|R| \leq 2$, so without loss of generality $c/\theta = \{c\}$. Let $K = T \setminus \{c\}$. Then $\mathbf{K}/\theta \in \mathbb{HS}(\mathbf{L})$, and thus satisfies (W). Therefore $\mathbf{R} \cong C(\mathbf{K}/\theta)$ satisfies (W). □

While Theorem 13 is useful for providing examples, using SIW, it is also interesting to see what happens when $\mathbf{L}$ is not SIW. The crucial observation is that if $\theta \in \operatorname{Con} \mathbf{L}$ is 0-separating and 1-separating, then θ extends to a congruence on $C(\mathbf{L})$, but otherwise not. And *otherwise not* in the worst possible way, as the congruence generated by θ on

$C(\mathbf{L})$ has $|C(\mathbf{L})/\operatorname{Cg}(\theta)| \leq 2$, depending on the number of atoms and coatoms of $\mathbf{L}$.

For example, $\mathbf{L}_{15}$ (see Figure 2) is IW, but $\mathbf{H}[m] \leq C(\mathbf{L}_{15})$, so $C(\mathbf{L}_{15})$ is not IW. On the other hand, our comments about $\mathbf{H}[m]$ apply equally to $C(\mathbf{H}[m])$. The lattice $C(\mathbf{H}[m])$ is s.i. and satisfies (W), but $C(\mathbf{H}[m])/\mu = C(\mathbf{H})$ is s.i. and fails (W). Therefore $\mathbb{V}(C(\mathbf{H}))$ is primitive by an argument using Theorem 26, while $\mathbb{V}(C(\mathbf{H}[m]))$ is not primitive by Corollary 5.

Theorem 14. *Let $\mathbf{L}$ be a finite lattice.*

(1) *$N(\mathbf{L})$ is s.i. iff $\mathbf{L}$ is s.i.*
(2) *$N(\mathbf{L})$ satisfies (W) iff $\mathbf{L}$ satisfies (W).*

Moreover, $N(\mathbf{L})$ is inherently Whitman if and only if $\mathbf{L}$ is inherently Whitman, and the same for SIW.

Starting with $\mathbf{2}$, this gives us the sequence of lattices $\mathbf{L}_6^k$ in Figure 3. But you can start with any inherently Whitman s.i. lattice.

Proof. (1) Writing $\mathbf{L} = [0_L, 1_L]$, we see that collapsing any of the intervals $[1, i]$, $[c, i]$, $[z, c]$ $[z, 0]$ collapses all of L. On the other hand, an interval in $\mathbf{L}$ can be collapsed without affecting any of those.

(2) is obvious.

Consider a sublattice $\mathbf{T} \leq N(\mathbf{L})$. We want to show that if $\mathbf{L}$ is inherently Whitman, then any s.i. lattice in $\mathbb{H}(\mathbf{T})$ satisfies (W). The interesting case is when $c \in T$ and $T \cap L \neq \varnothing$, otherwise we are back to $\mathbb{HS}(\mathbf{L})$. Put $K = T \cap L$. Then the preceding remarks apply to the sublattice $K \cup \{i, c, z\}$. The new s.i.'s we get are of the form $N(\mathbf{R})$ with $\mathbf{R} \in \mathbb{HS}(\mathbf{K})$ and s.i. Such an $\mathbf{R}$ satisfies (W) as it is s.i. and in $\mathbb{HS}(\mathbf{L})$, and $\mathbf{L}$ was assumed to be inherently Whitman. □

The remaining constructions U, D, Q, and Q^* require more care.

Theorem 15. *Let $\mathbf{L}$ be a finite lattice, and let $a < 1_L$ be in $\mathrm{J}(\mathbf{L})$.*

(1) *If $1_L \not\succ a$ and $\mathbf{L}$ is s.i., then $U(\mathbf{L}, a)$ is s.i.*
(2) *Assume that a satisfies*

$$(\digamma) \quad \text{if } a \leq x \vee y \text{ nontrivially, then } x \vee y = 1_L\,.$$

Then $U(\mathbf{L}, a)$ satisfies (W) iff $\mathbf{L}$ satisfies (W).

Moreover, if $\mathbf{L}$ is strongly inherently Whitman and a satisfies $(\digamma)$, and every $b < a$ is join prime, then $U(\mathbf{L}, a)$ is strongly inherently Whitman.

Note that, in particular, the theorem applies when a satisfies $(\digamma)$ and $a \succeq 0_L$.

Proof. (1) If there exists b with $a < b < 1_L$, then $\mathbf{L}$ s.i. implies $U(\mathbf{L}, a)$ s.i., because the intervals $[a, c]$, $[c, i]$, $[1_L, i]$ collapse $[b, 1_L]$. The converse is false: $U(\mathbf{2} \times \mathbf{2}, 0)$ is $\mathbf{L}_4$, which is s.i. Also, the intermediate step of the Q or Q^* construction may not be s.i.

(2) $U(\mathbf{L}, a)$ satisfies (W) iff $\mathbf{L}$ satisfies (W). The new nontrivial meets in $U(\mathbf{L}, a)$ are of the form $c \wedge u$ for some $u \in L$. But $c \wedge u = a \wedge u$, and we can apply (W) in $\mathbf{L}$, and apply (F) to $a \wedge u \leq x \vee y$.

For the last claim, assume

- $\mathbf{L}$ is strongly inherently Whitman,
- $\mathbf{T} \leq U(\mathbf{L}, a)$,
- $\mathbf{R} = \mathbf{T}/\varphi$.

The hypothesis that elements strictly below a are join prime will be added when needed. We want to show that $\mathbf{R}$ satisfies (W). We may assume that $\mathbf{R} \notin \mathbb{HS}(\mathbf{L})$ or else the conclusion is immediate.

Consider $\mathbf{T}$. Without loss of generality $c \in T$ (else $\mathbf{T} \leq \mathbf{L} + \mathbf{1}$) and set $K = T \cap L$. Let $a' = 1_K \wedge c = 1_K \wedge a$, which is j.i. since $\mathbf{L}$ satisfies (W). Thus $\mathbf{T} = U(\mathbf{K}, a')$, but for $\mathbf{T}$ to satisfy (W) we need that a' satisfies (F). Suppose $a' = 1_K \wedge a \leq x \vee y$ nontrivially in $\mathbf{T}$. Apply (W). "Nontrivial" means that $a' \nleq x$ and $a' \nleq y$, while $1_K \leq x \vee y$ is the desired conclusion. The other possibility is $a \leq x \vee y$; since a satisfies (F) that implies $x \vee y = 1_L \geq 1_K$.

Now consider $\mathbf{R} = \mathbf{T}/\varphi$ for any congruence φ. If $(a', b) \in \varphi$ for some $b > a'$, then $(a', 1_K) \in \varphi$. In this case $\mathbf{R} \cong \mathbf{K}/\varphi$ or $\mathbf{R} \cong \mathbf{K}/\varphi + \mathbf{1}$, which by assumption satisfy (W). Thus $[a'] \subseteq \downarrow a'$, whence $\mathbf{R} = U(\mathbf{K}/\varphi, [a'])$.

Let $\beta(a)$ be the least element in the congruence class $[a']$. To apply (2), we need that $\beta(a)$ satisfies (F). If $\beta(a) = a$, then it satisfies (F) by assumption, while if $\beta(a) < a$, then it is join prime, which is stronger than (F). Note that if b is join prime in $\mathbf{L}$, then it is join prime in any sublattice containing b, and if b is join prime and the least element of its congruence class, then $[b]$ is join prime in the quotient lattice. □

The wing-down construction $D(\mathbf{L}, a)$ has dual properties. The $Q(\mathbf{L}, a)$ and $Q^*(\mathbf{L}, a)$ constructions combine U and D.

Theorem 16. *Assume $\mathbf{L}$ is strongly inherently Whitman and a satisfies (F), and that every element $b < a$ is join prime. If $a > 0$, also assume that a is meet irreducible. Then $Q(\mathbf{L}, a)$ is strongly inherently Whitman.*

The condition that a be meet irreducible is to ensure that the dual of (F) holds for c in $U(\mathbf{L}, a)$. The proof of the theorem is straightforward.

We can get lots of sequences with alternating up-downs, satisfying $a \succeq 0$ for the up steps and $a \preceq 1$ dually for the down steps. The chain

$\mathbf{L}_9^k$ in the middle of Figure 3 comes this way, starting with $\mathbf{L}_9$, which in turn is $Q(\mathbf{N}_5, a)$. For example, $\mathbf{L}_9^{2n} = (DU)^n(\mathbf{L}_9)$. Hence $\mathbf{L}_9^k$ is a sequence of s.i. SIW lattices.

The lattice $\mathbf{L}_9$ illustrates why Ježek and Slavík [12], in characterizing s.i. sublattices of a free lattice, included the Q and Q^* constructions.

Lemma 17. *Let* $\mathbf{L}$ *be a finite s.i. lattice and* $0 < a \prec 1$*. Then* $U(\mathbf{L}, a)$ *is not subdirectly irreducible, but* $Q(\mathbf{L}, a) = D(U(\mathbf{L}, a), c)$ *is s.i.*

The lemma applies to the SIW s.i. lattices $\mathbf{N}_5$, $\mathbf{L}_4$, $\mathbf{L}_5$, $\mathbf{L}_6^k$, and the covers $\mathbf{K}_1$ and $\mathbf{K}_3$ of $\mathbf{L}_4$, and their duals, amongst others.

The lattice $\mathbf{K}_2$ generating a cover of $\mathbb{V}(\mathbf{L}_4)$ is an interesting case, as it is SIW s.i. and has two candidates for a: $0 \prec a_1 \prec a_2 \prec 1$. Both a_1 and a_2 satisfy the conditions of Theorem 15, whence both $Q(\mathbf{K}_2, a_1)$ and $Q(\mathbf{K}_2, a_2)$ are s.i. and SIW. But the first step, $U(\mathbf{K}_2, a_1)$ is s.i., while $U(\mathbf{K}_2, a_2)$ is not. This illustrates Lemma 17.

Figure 10 shows the SIW lattice $D(U(\mathbf{L}_1, a), c)$. This alternating pattern can be continued *ad infinitum* to form an infinite chain of SIW varieties. Because $a \not\prec 1$ in $\mathbf{L}_1$, the first step: $U(\mathbf{L}_1, a)$ is s.i., and we don't need Lemma 17. Note that $\mathbf{L}_4 \leq U(\mathbf{L}_1, a)$, so that $\mathbb{V}(U(\mathbf{L}_1, a)) \geq \mathbb{V}(L_1) \vee \mathbb{V}(\mathbf{L}_4)$; in particular, it does not cover $\mathbb{V}(\mathbf{L}_1)$ (which we already knew).

The lattices $\mathbf{L}_{12}^k$ of Figure 3 are obtained by applying the $U(\mathbf{L}, a)$ construction repeatedly, starting with $\mathbf{L}_{12}$. But this sequence does not satisfy the restriction that elements $b < a$ be join prime, so to show that this gives a sequence of inherently Whitman varieties, we must appeal to the fact that it is a chain in Λ containing only those lattices, along with $\mathbf{2}$ and $\mathbf{N}_5$, as s.i.'s. Indeed, the lattice $\mathbf{L}_{12}^k$ are not SIW for $k \geq 3$, as illustrated in Figure 11.

We would like a version of Theorem 15 for IW. The example in Figure 12 shows that would be difficult. Let $\mathbf{L}$ be the lattice $\downarrow 1_L$ in the figure. Then $\mathbf{L}$ satisfies (W) and is semidistributive (so projective), and is inherently Whitman. It is not s.i. The elements x_7 and its lower cover x_6 are join prime, which is stronger than (F). We will calculate that $\mathbf{L}$ is IW, but $U(\mathbf{L}, x_7)$ is not. At the price of losing $(\mathrm{SD}_\vee)$ in the extended lattice, we could show the same for $U(\mathbf{L}, x_6)$. The calculation uses the following lemma [7, 18]; cf. Section III.2 of [6].

Recall Day's D-relation on $\mathrm{J}(\mathbf{L})$: $p \mathrel{\mathrm{D}} q$ if q is a member of a minimal (nonrefinable) nontrivial join cover of p, i.e., $p \leq \bigvee Q$ minimally and $q \in Q$. For an element $u \in \mathrm{J}(\mathbf{L})$, let $B(u) = \{u\} \cup \{q : u \mathrel{\mathrm{D}^n} q$ for some $n \geq 1\}$. Let $\mathrm{L}(u)$ be the join semilattice with 0 of $\mathbf{L}$ generated by $B(u)$.

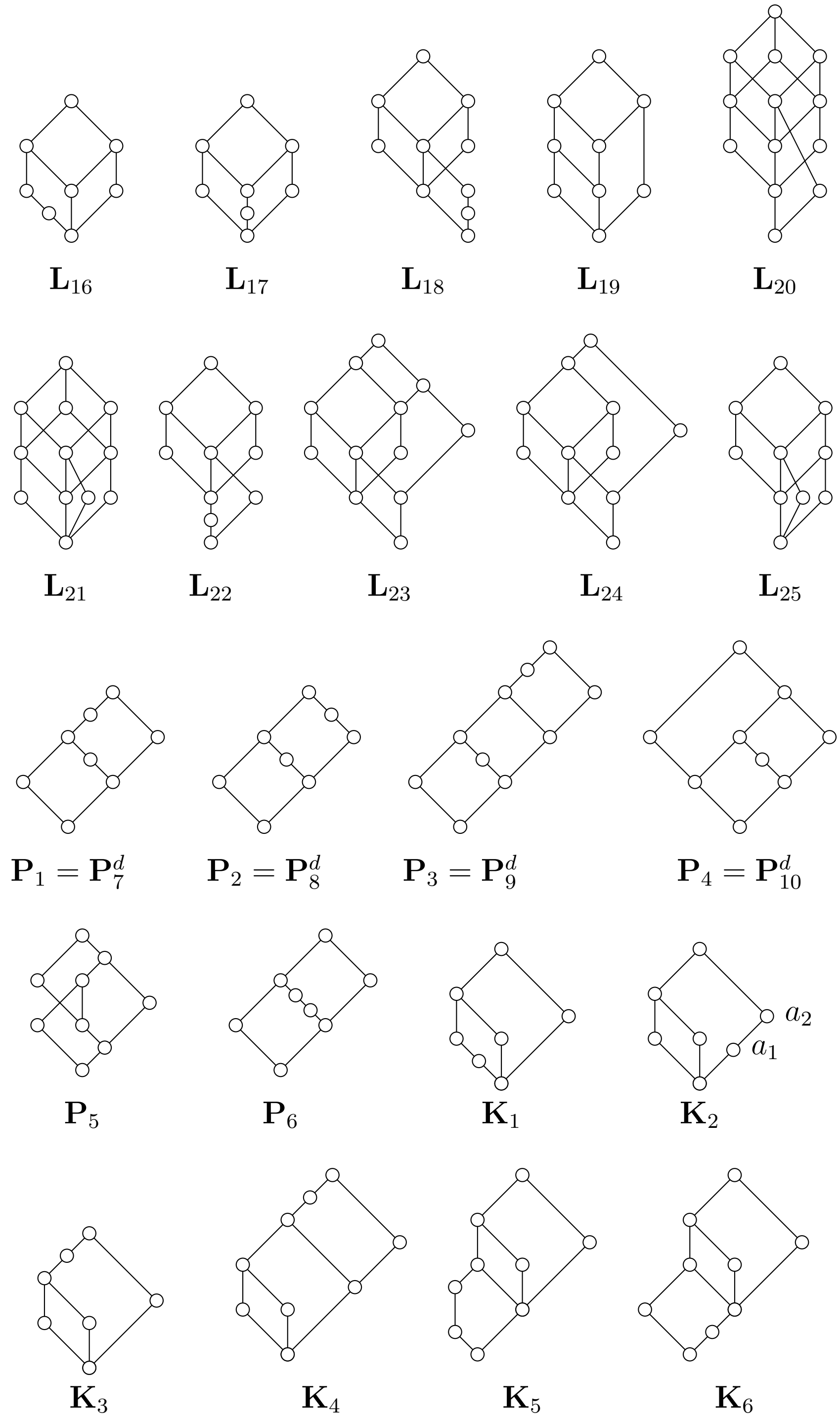


FIGURE 9. $\mathbf{L}_{16}$–$\mathbf{L}_{25}$, $\mathbf{P}_1$–$\mathbf{P}_{10}$, and $\mathbf{K}_1$-$\mathbf{K}_6$ generate all join irreducible covers of $\mathbb{V}(\mathbf{L}_1)$, $\mathbb{V}(\mathbf{L}_3)$, and $\mathbb{V}(\mathbf{L}_4)$ respectively.

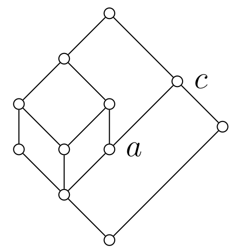


FIGURE 10. The strongly inherently Whitman lattice $D(U(\mathbf{L}_1, a), c)$.

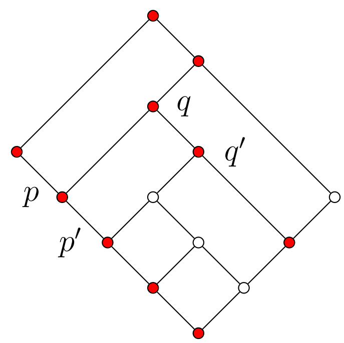


FIGURE 11. The lattice $\mathbf{L}_{12}^3$ is not SIW. The congruence that collapses (p, p') and (q, q') on the red sublattice does not satisfy (W).

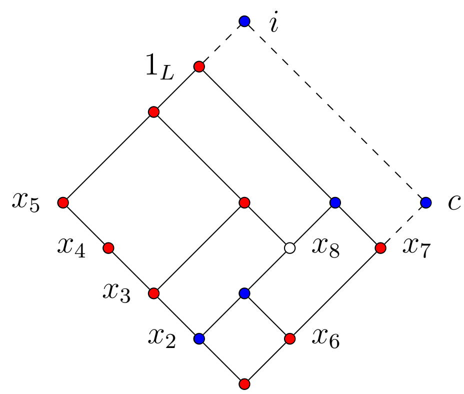


FIGURE 12. $\mathbf{L}$ satisfies (W) and is IW, but $U(\mathbf{L}, x_7)$ is not IW. The congruence $\psi(x_5)$ on $U(\mathbf{L}, x_7)$ collapsing x_8 to its lower cover has $\mathbf{L}/\psi(x_5)$ s.i. but not satisfying (W).

Meanwhile, still with u join irreducible, there is a unique largest congruence $\psi(u)$ on $\mathbf{L}$ that does not contain (u, u_*), making $\mathbf{L}/\psi(u)$ s.i. with (u, u_*) its critical quotient.

Lemma 18. *Let* $\mathbf{L}$ *be a finite lattice. The s.i. homomorphic images of* $\mathbf{L}$ *are* $\mathbf{L}/\psi(u)$ *for* $u \in \mathrm{J}(\mathbf{L})$*. Moreover,* $\mathbf{L}/\psi(u) \cong \mathrm{L}(u)$*.*

For the lattice $\mathbf{L}$ in Figure 12, without the wing, $\mathrm{J}(\mathbf{L}) = \{x_j : 2 \leq j \leq 8\}$. The minimal nontrivial join covers are $x_4 \leq x_3 \vee x_7$, $x_5 \leq x_3 \vee x_7$, $x_5 \leq x_4 \vee x_6$, $x_8 \leq x_2 \vee x_7$. That makes $B(x_5) = \{x_j : 3 \leq j \leq 7\}$, and $\mathrm{L}(x_5)$ the subsemilattice indicated by the red nodes, which is isomorphic to $\mathbf{L}_9$.

But in $\mathbf{L}' = U(\mathbf{L}, x_7)$, there is a new minimal join cover $x_j \leq x_2 \vee c$ for $j \neq 6, 7, 8$. Now $B'(u) = B(u) \cup \{x_2, c\}$, and $\mathrm{L}'(u)$ includes everything except x_8 (red and blue nodes). In $\mathrm{L}'(u)$, the single congruence class (x_8, x_{8*}) represents a doubly reducible element, and (W) fails. Thus $U(\mathbf{L}, x_7)$ is not IW.

That still leaves the possibility that if $\mathbf{L}$ is a finite, subdirectly irreducible, semidistributive lattice satisfying (W), then $\mathbb{V}(\mathbf{L})$ is primitive. This is true for $|L| \leq 10$, and examples show that you need both semidistributive laws, $\mathrm{SD}_\wedge$ and $\mathrm{SD}_\vee$. But it is still false.

Example 19.

The lattice in Figure 13 is a sublattice of Ježek and Slavík's (s.i., projective) lattice $\mathbf{C}_1$ from [12], obtained by removing one doubly irreducible element. It has a homomorphic image $\mathbf{K}$, collapsing w and $w \vee t$ down, which fails (W): $x \wedge y \leq t \vee v$ in $\mathbf{K}$. Moreover, $\mathbf{K}$ is s.i., with $[t \vee v, x]$ as the critical quotient. This can be seen from the minimal nontrivial join covers in $\mathbf{L}$:

$$x \leq y \vee t$$
$$x \leq z \vee t$$
$$y \leq u \vee z$$
$$u \leq t \vee v$$

while w is not part of any minimal join cover. Of course $\mathbf{K} \not\leq \mathbf{C}_1$ since the latter satisfies (W), so $\mathbf{C}_1$ is not primitive.

4. A great multitude of sequences of SIW lattices

Now it is clear how to make manifold sequences of SIW lattices. Start with a finite, s.i., SIW lattice. Any of the following from Jipsen and Rose [14] will work: $\mathbf{M}_k$, $\mathbf{N}_5$, $\mathbf{L}_j$ for $1 \leq j \leq 14$ or $j = 16, 17, 19$, $\mathbf{P}_j$ for $1 \leq j \leq 4$ or $j = 6$, $\mathbf{K}_j$ for $1 \leq j \leq 5$, $\mathbf{T}_3$, $\mathbf{V}_j$ for $j = 1, 2, 6, 8$ or their duals; see Figures 2 and 9. Then apply, in any order, a sequence of operations from $\{C, N, U, D, Q, Q^*\}$ as long as the conditions of Theorem 15 hold for U, or its dual for D. Some care must be exercised

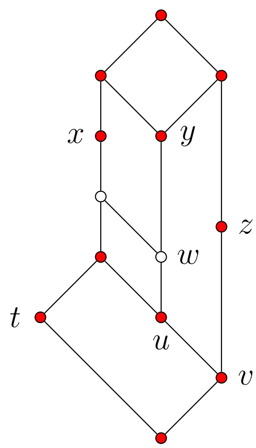


FIGURE 13. Example 19: $\mathbf{L}$ is semidistributive and satisfies (W), but has a homomorphic image (indicated by red nodes) that is s.i. and fails (W).

with the conditions for consecutive steps UU and dually consecutive steps DD, to be sure that the theorem applies (as we have seen with the failure of $\mathbf{L}_{12}^k$ to be SIW).

Recall that $\mathbf{M}_\omega$ also generates a primitive lattice variety. An analogous statement holds for our sequences.

Theorem 20. *Assume that $\mathbf{S}_k$ $(k \in \omega)$ form a chain of IW lattices:*

$$\mathbf{S}_0 \leq \mathbf{S}_1 \leq \mathbf{S}_2 \leq \dots$$

Let $\mathbf{S}_\omega = \bigcup_{k\in\omega} \mathbf{S}_k$. If $\mathbb{V}(\mathbf{S}_\omega)$ is locally finite, then it is a primitive lattice variety.

Proof. Fix $n \geq 0$, and let $\mathbf{F}_k$ be the n-generated free lattice in $\mathbb{V}(\mathbf{S}_k)$. Then

$$\mathbf{F}_0 \leq \mathbf{F}_1 \leq \mathbf{F}_2 \leq \dots$$

and since $\mathbb{V}(\mathbf{S}_\omega)$ is locally finite, there exists N such that $\mathbf{F}_N = \mathbf{F}_{N+1} = \dots = \mathbf{F}_\omega$. Hence every n-generated lattice in $\mathbb{V}(\mathbf{S}_\omega)$ is in $\mathbb{V}(\mathbf{S}_N)$. In particular, every finitely generated s.i. in $\mathbb{V}(\mathbf{S}_\omega)$ satisfies (W), and so is projective in the class of finite lattices. We can apply Theorem 1 to conclude that $\mathbb{V}(\mathbf{S}_\omega)$ is primitive. $\square$

Having to check that $\mathbb{V}(\mathbf{S}_\omega)$ is locally finite is inconvenient, but it is clearly necessary. All the lattices $\mathbf{L}_{12}^k$ $(k \geq 0)$ of Figure 3 are 4-generated, by the 2 atoms and 2 doubly irreducible elements. Thus $\mathbb{V}(\mathbf{L}_{12}^\omega)$ is not locally finite, and Theorem 20 does not apply. Note that $\mathbf{L}_{12}^k = U^k(\mathbf{L}_{12}, *)$.

But for our other constructions, excepting U and D, the number of generators increases. Let $\nu(\mathbf{L})$ denote the minimum number of generators of a finite lattice $\mathbf{L}$.

Lemma 21. *The constructions of* Section 3 *satisfy the following.*

(1) $\nu(C(\mathbf{L})) = \nu(\mathbf{L}) + 1$
(2) $\nu(N(\mathbf{L})) = \nu(\mathbf{L}) + 1$
(3) $\nu(U(\mathbf{L}, 0)) = \nu(\mathbf{L}) + 1$
(4) $\nu(D(\mathbf{L}, 1)) = \nu(\mathbf{L}) + 1$

Moreover, for the standard construction using atoms and coatoms, and $k \geq 1$*,*

(5) $\nu(\mathbf{L}) + k - 1 \leq \nu(Q^k(\mathbf{L}, a)) \leq \nu(\mathbf{L}) + k$
(6) $\nu(\mathbf{L}) + k - 1 \leq \nu(Q^{*k}(\mathbf{L}, a)) \leq \nu(\mathbf{L}) + k$

The proofs are routine calculations. For (5) and (6), the first application of Q or Q^* may not increase the number of generators, but those thereafter do. (Consider $\mathbf{Q}(\mathbf{M}_3)$ versus $\mathbf{Q}(\mathbf{L}_1)$.)

It follows from the lemma that $\mathbb{V}(\mathbf{S}_\omega)$ is locally finite for a sequence obtained using the constructions listed in (1)–(6), so that Theorem 20 applies. Though that theorem only uses that the lattices in the sequence be IW, most of our constructions require SIW, and the sequences constructed in the next theorem will consist of SIW lattices.

Corollary 22. *There are* $2^{\aleph_0}$ *primitive lattice varieties.*

Proof. For each sequence $\mathbf{s} = s_1 s_2 s_3 \cdots \in 2^{\mathbb{N}}$ we produce a sequence $\mathbf{S}_k$ ($k \in \omega$) of lattices to which Theorem 20 applies, in such a way that distinct sequences produce distinct primitive varieties $\mathbb{V}(\mathbf{S}_\omega)$. Let $\mathbf{S}_0 = \mathbf{M}_3$, and recursively

$$\mathbf{S}_k = \begin{cases} Q(\mathbf{S}_{k-1}, a_{k-1}) & \text{if } s_k = 1, \\ Q^*(\mathbf{S}_{k-1}, a_{k-1}) & \text{if } s_k = 0, \end{cases}$$

where a_{k-1} is chosen as indicated in Figure 14; cf. the sequence $\mathbf{L}_9^k$ in Figure 3. To apply the Q-construction, Theorem 16 or its dual, one must check that each a_k is doubly irreducible, with elements $b < a_k$ join prime and $b > a_k$ meet prime. Each $\mathbf{S}_k$ is s.i. by Lemma 17. To see that distinct sequences give distinct varieties, consider another sequence $\mathbf{t} \in 2^{\mathbb{N}}$ giving a sequence of lattices $\mathbf{T}_\ell$ ($\ell \in \omega$). First observe that because every $\mathbf{S}_k$ is SIW, $\mathbf{S}_k \in \mathbb{V}(\mathbf{T}_\ell)$ implies $\mathbf{S}_k \leq \mathbf{T}_\ell$. It remains to check that $\mathbf{S}_k \leq \mathbf{T}_\ell$ iff $k \leq \ell$ and $s_1 \ldots s_k$ is an initial segment of $\mathbf{t}$. Both these statements can be proved by induction. □

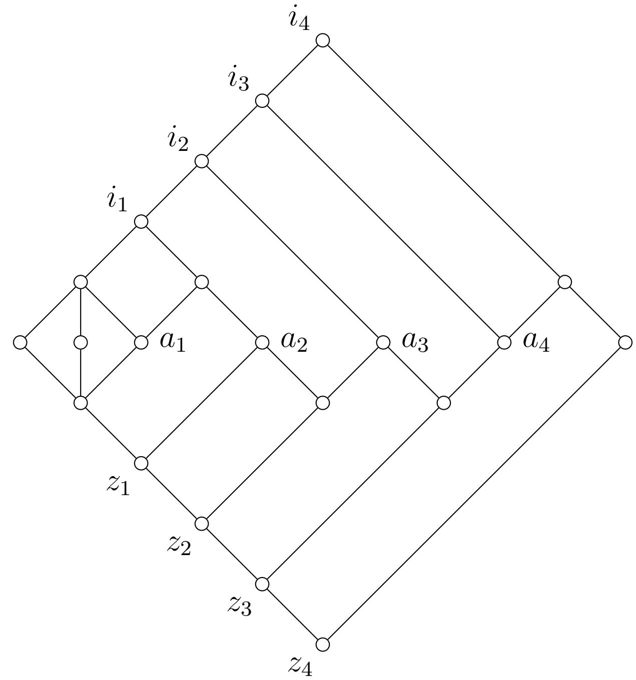


FIGURE 14. The construction of Corollary 22 for the sequence 1001: $QQ^*Q^*Q((((\mathbf{M}_3, a_1), a_2), a_3), a_4)$. $\mathbf{S}_k$ is the interval $[z_k, i_k]$.

5. Some primitive lattice varieties failing (W)

Theorem 2 states that if $\mathbf{L}$ is IW, then $\mathbb{V}(\mathbf{L})$ is primitive. *Is it possible for a lattice that fails (W) to generate a primitive variety?* Failing (W) says that $\mathbf{L}$ is not projective in the class of all finite lattices, when all you need is that it be weakly projective in the variety it generates. And indeed, in this section we will see some primitive varieties generated by lattices failing (W).

As an immediate consequence of Theorem 3 we have the following.

Lemma 23. *If $\mathbf{L}$ is s.i. and I is a (W)-failure interval, then the variety $\mathbb{V}(\mathbf{L}[I])$ is not primitive.*

Proof. Let $\mathbf{L}$ be s.i. with a (W)-failure interval I. Then $\mathbf{L}$ is a homomorphic image of $\mathbf{L}[I]$, but not a sublattice (Theorem 6). That says that $\mathbb{V}(\mathbf{L}[I])$ is not primitive. □

It still seems possible that $\mathbb{V}(\mathbf{L})$ might be primitive, though, and in fact that can happen. By the lemma, a necessary condition is that

$\mathbb{V}(\mathbf{L}) \neq \mathbb{V}(\mathbf{L}[I])$, i.e., $\mathbf{L}[I] \notin \mathbb{V}(\mathbf{L})$. That is what goes wrong with $\mathbb{V}(\mathbf{M}_{3,3})$. So we seek lattices $\mathbf{L}$ such that

- $\mathbf{L}$ is s.i. with a (W)-failure interval I,
- $\mathbf{L}[I] \notin \mathbb{V}(\mathbf{L})$, for example, $\mathbf{L}[I]$ could be s.i.,
- $\mathbf{L}$ is weakly projective in $\mathbb{V}(\mathbf{L})$,
- every s.i. in $\mathbb{HS}(\mathbf{L}) \setminus \{\mathbf{L}\}$ is also weakly projective in $\mathbb{V}(\mathbf{L})$, for example, by satisfying (W).

Let us first assemble some tools we will use, and then set up a working algorithm, which while not complete will decide whether $\mathbb{V}(\mathbf{L})$ is primitive in many cases. We will give three types of concrete examples. After that, in Section 6, we give the results of the algorithm for all s.i. lattices with $|L| \leq 9$.

A useful observation is the *splitting lemma*.

Lemma 24. *If the interval $I = [s, t]$ of $\mathbf{L}$ is such that $L = \uparrow s \cup \downarrow t$, then $\mathbf{L}[I] \leq \mathbf{L} \times \mathbf{2}$, whence $\mathbf{L}[I] \in \mathbb{V}(\mathbf{L})$.*

For example, this is how you see that $\mathbf{M}_{3,3}$, $\mathbf{K}_6$, and $\mathbf{V}_4$ do not generate primitive varieties.

Many of the smaller lattices we encounter have a single (W)-failure. While we do not have a general tool for that case, the following restricted version will serve our purposes.

First we identify the condition

> $\varpi(\mathbf{L}, \mathbf{M})$: If $\mathbf{K}$ is a finite lattice and $f : \mathbf{K} \twoheadrightarrow \mathbf{L}$ a surjective homomorphism, then either $\mathbf{L} \leq \mathbf{K}$ or $\mathbf{M} \leq \mathbf{K}$.

If $\varpi(\mathbf{L}, \mathbf{M})$ holds, then $\mathbf{L}$ is weakly projective in any variety that contains $\mathbf{L}$ but not $\mathbf{M}$. (Of course, that is vacuous if $\mathbf{M} \in \mathbb{V}(\mathbf{L})$.) As with the distinction between projectivity and weak projectivity, we actually wind up using a slightly stronger version,

> $\varpi'(\mathbf{L}, \mathbf{M})$: If $\mathbf{K}$ is a finite lattice and $f : \mathbf{K} \twoheadrightarrow \mathbf{L}$ a surjective homomorphism, then either there is a retraction $g : \mathbf{L} \leq \mathbf{K}$, or $\mathbf{M} \leq \mathbf{K}$.

Lemma 25. *Let $\mathbf{L}$ be a finite lattice with only one (W)-failure, say the interval $I = [s, t]$, such that $s \preceq t$. Further assume the following.*

(1) *If $s = t$, then s has exactly 2 upper covers, a and b, and $s = a \wedge b$ is the unique irredundant proper meet decomposition of s.*
(2) *If $s \prec t$, then s has a unique upper cover a with $a \nleq t$, such that $s = a \wedge t$ and $s = a \wedge t^*$ are the only irredundant proper meet decompositions of s, where t^* is the unique upper cover of t.*

Then $\varpi'(\mathbf{L}, \mathbf{L}[I])$ holds.

While the conditions are restrictive, they hold in a number of small examples.

Proof. As in the hypothesis of $\varpi'(\mathbf{L}, \mathbf{L}[I])$, let $\mathbf{K}$ be a finite lattice and $f : \mathbf{K} \twoheadrightarrow \mathbf{L}$ a surjective homomorphism. Again we start with $g_0(x)$ being the largest pre-image of x, and form the sequence of maps g_k until it terminates in a join-preserving transversal g. Since g_0 preserves meets, g will preserve some meets, namely, $g(x \wedge y) = g(x) \wedge g(y)$ whenever $x \wedge y \notin I$ by Lemma 4(iv).

To simplify notation, write $\overline{x} = g(x)$ for $x \in L$.

Now we distinguish two cases. First assume $s = t$ is a doubly reducible element, with $s = a \wedge b = \bigvee C$ nontrivially. Our extra assumption (1) implies that if $D \subseteq L$ is a subset such that $s = \bigwedge D$ nontrivially, then $a, b \in D$. Hence if $f(\bigwedge_{d \in D} \overline{d}) = s$, then $\bigwedge_{d \in D} \overline{d} = \overline{a} \wedge \overline{b}$.

If $\overline{a} \wedge \overline{b} = \overline{s} = \bigvee_{c \in C} \overline{c}$, then g is an embedding of $\mathbf{L}$ into $\mathbf{K}$. So assume $\overline{a} \wedge \overline{b} > \bigvee_{c \in C} \overline{c}$.

Let $m_0 = \overline{s} = \bigvee_{c \in C} \overline{c}$ and $m_1 = \overline{a} \wedge \overline{b}$, both of which are pre-images of s in $\mathbf{K}$. To get an embedding of $\mathbf{L}[s]$ into $\mathbf{K}$, we need to show that if $u \nleq s$, then $\overline{u} \vee m_0 \geq m_1$ in $\mathbf{K}$, and if $v \ngeq s$, then $\overline{v} \wedge m_1 \leq m_0$ in $\mathbf{K}$. The first is because g is a join-preserving transversal. Thus $u \vee s > s$ implies $\overline{u} \vee \overline{s} = \overline{u \vee s} > \overline{s}$, whence $\overline{u} \vee m_0 \geq \overline{a}$ or $\overline{u} \vee m_0 \geq \overline{b}$. In either event $\overline{u} \vee m_0 \geq m_1$. Likewise, suppose $v \ngeq s$. Then because $g(\bigwedge T) = \bigwedge g(T)$ except when $\bigwedge T = s$, we have $\overline{v} \wedge m_1 = \overline{v} \wedge \overline{a} \wedge \overline{b} = \overline{v \wedge s} < \overline{s} = m_0$ since $v \wedge s < s$, whence $\overline{v} \wedge m_1 < \overline{s} = m_0$.

Next we consider case (2) when $s \prec t$, with say $t = \bigvee C$. First note that since t is a proper join and $[s, t]$ is the only (W)-failure in $\mathbf{L}$, the element t must be meet irreducible. Thus it has a unique upper cover t^*. Assumption (2) is that if $D \subseteq L$ is such that $s = \bigwedge D$ nontrivially, then $a \in D$ and either t or t^* is in D. Hence if $f(\bigwedge_{d \in D} \overline{d}) = s$, then $\bigwedge_{d \in D} \overline{d} = \overline{a} \wedge \overline{t}$ or $\bigwedge_{d \in D} \overline{d} = \overline{a} \wedge \overline{t}^*$.

Now $\overline{s}$ has 2 upper covers in $g(L)$, which are $\overline{a}$ and $\overline{t}$. Moreover, since s is meet reducible and $[s, t]$ is the only (W)-failure, s is join irreducible. Thus, replacing the original $g(s)$ if necessary, we may assume that $\overline{s} = \overline{a} \wedge \overline{t}$, without changing the fact that g is join-preserving.

If $\overline{a} \wedge \overline{t}^* \leq \overline{t}$, then $\overline{a} \wedge \overline{t}^* = \overline{s}$, and g embeds $\mathbf{L}$ into $\mathbf{K}$. So assume $\overline{s} = \overline{a} \wedge \overline{t} < \overline{a} \wedge \overline{t}^*$. Consider the following elements in $\mathbf{K}$.

$$m_0 = \overline{s} \qquad\qquad n_0 = \overline{t}$$

$$m_1 = \overline{a} \wedge \overline{t}^* \qquad\qquad n_1 = \overline{t} \vee m_1$$

It is straightforward to calculate that m_0, m_1, n_0, n_1, $\overline{t}^*$ form a small sublattice of $\mathbf{K}$ isomorphic to $\mathbf{2} \times \mathbf{2} + \mathbf{1}$. We claim that $g(L) \cup \{m_1, n_1\}$

is isomorphic to $\mathbf{L}[I]$. It remains to check that $x \wedge \overline{u}$ and $x \vee \overline{u}$ behave correctly for $u \in L$ and $x \in \{m_0, m_1, n_0, n_1, \overline{t}^*\}$. This is tedious but routine, using the covering relations in $\mathbf{L}$: if $u \geq s$, then $u = s$ or $u \geq a$ or $u = t$ or $u \geq t^*$. $\square$

Lemma 25 and the condition $\varpi(\mathbf{L}, \mathbf{M})$ are clearly designed for the case when $\mathbf{L}$ has only one (W)-failure. If there were two, say I and J, then $\mathbf{K}$ could double I or J or both, for a minimum of 3 possibilities. In Lemma 32, as part of showing that $\mathbf{U}_4$ with many doubly reducible elements generates a primitive variety, we will get around the problem by excluding $\mathbf{L}_{15}$ from $\mathcal{V}$. Lemma 32 shows that $\varpi'(\mathbf{U}_4, \mathbf{L}_{15})$ holds.

Here is our working test to answer the question: *Is $\mathcal{V}$ primitive?* We are thinking $\mathcal{V} = \mathbb{V}(\mathbf{L})$, but it could be any locally finite variety.

(A) If $\mathcal{V}$ is IW, then YES.

(B) If $\mathcal{V}$ contains a finite lattice $\mathbf{L}$ that satisfies (W) but $\mathbf{L}$ is not IW, i.e., there exists a s.i. $\mathbf{K} \in \mathbb{HS}(\mathbf{L})$ that fails (W), then NO.

(C) If there is a s.i. $\mathbf{K}$ in $\mathcal{V}$ which has a (W)-failure I such that $\mathbf{K}[I] \in \mathcal{V}$, then NO.

(D) If for every s.i. $\mathbf{K}$ in $\mathcal{V}$, either $\mathbf{K}$ satisfies (W) or there is a single (W)-failure interval I of $\mathbf{K}$, such that the property $\varpi(\mathbf{K}, \mathbf{K}[I])$ holds and $\mathbf{K}[I] \notin \mathcal{V}$, then YES.

(E) If for every s.i. $\mathbf{K}$ in $\mathcal{V}$, either $\mathbf{K}$ satisfies (W) or there is a lattice $\mathbf{M} \notin \mathcal{V}$ such that the property $\varpi(\mathbf{K}, \mathbf{M})$ holds, then YES.

Note that in (D), $\mathbf{K}[I] \notin \mathcal{V}$ could hold either because (I) $\mathbf{K}[I]$ is s.i. and not in $\mathcal{V}$, or (II) $\mathbf{K}[I]$ is subdirectly reducible but one of its factors is not in $\mathcal{V}$. Congruence distributivity helps here, because it makes subdirect decompositions unique.

What we *don't* know is whether there might not exist a s.i. lattice $\mathbf{K} \in \mathcal{V}$ with a (W)-failure interval J such that $\mathbf{K}[J] \notin \mathcal{V}$, but $\varpi(\mathbf{K}, \mathbf{K}[J])$ does not hold, and in fact there is a lattice $\mathbf{L}' \in \mathcal{V}$ with a surjective homomorphism $f : \mathbf{L}' \twoheadrightarrow \mathbf{K}$ while $\mathbf{K} \not\leq \mathbf{L}'$.

The examples below give lattices that fail (W) but generate a primitive variety. Those in the first example have a single doubly reducible element for their (W)-failure, while those in the second example have a proper (W)-failure interval. The third example contains 3 doubly reducible elements.

We have yet to find a lattice with more than 1 (W)-failure, including a proper one, that generates a primitive variety. The lattice $\mathbf{G}_6$ of Figure 20 is typical of what we encounter. It has a doubly reducible element m and a proper failure I. The lattice $\mathbf{G}_6[m]$ is not in $\mathbb{V}(\mathbf{G}_6)$, but the interval I splits the lattice, so $\mathbf{G}_6[I] \in \mathbb{V}(\mathbf{G}_6)$. The latter is enough to ensure that $\mathbb{V}(\mathbf{G}_6)$ is not primitive.

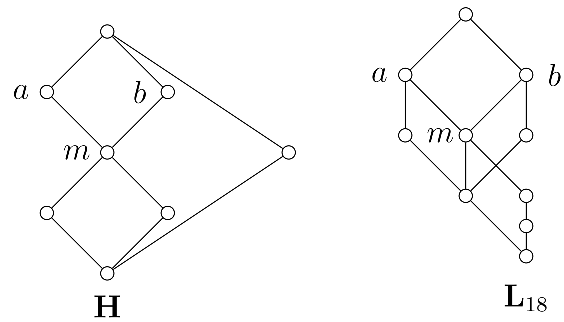


FIGURE 15. Two s.i. lattices that fail Whitman's condition (W) but generate a primitive lattice variety.

5.1. **First example.** In these lattices, the (W)-failure consists of a single doubly reducible element. As expected, we use Lemma 25.

Theorem 26. *Let* $\mathbf{L}$ *be a finite s.i. lattice with a doubly reducible element* m *satisfying the property of* Lemma 25(1). *If, in addition,*

(1) $\mathbf{L}[m] \notin \mathbb{V}(\mathbf{L})$, *and*

(2) *every s.i. lattice in* $\mathbb{HS}(\mathbf{L}) \setminus \{\mathbf{L}\}$ *is weakly projective in* $\mathbb{V}(\mathbf{L})$,

then $\mathbb{V}(\mathbf{L})$ *is a primitive lattice variety.*

In particular, (2) holds when those lattices satisfy (W).

Proof. By Theorem 1, we need to show that every s.i. in $\mathbb{V}(\mathbf{L})$ is weakly projective in $(\mathbb{V}(\mathbf{L}))_{\text{fin}}$. This is true for $\mathbf{L}$ since $\varpi(\mathbf{L}, \mathbf{L}[m])$ holds and $\mathbf{L}[m] \notin \mathbb{V}(\mathbf{L})$, and assumption (2) is that it holds for the other s.i. lattices in $\mathbb{V}(\mathbf{L})$. □

The theorem applies to our familiar example $\mathbf{H}$, as labelled in Figure 15. By Lemma 25, $\mathbf{H}$ is projective in any locally finite variety not containing $\mathbf{H}[m]$. The other s.i. lattices in $\mathbb{V}(\mathbf{H})$ are $\mathbf{2}$, $\mathbf{N}_5$, $\mathbf{L}_4$, $\mathbf{L}_5$, $\mathbf{K}_3$, and $\mathbf{K}_3^d$, all of which satisfy (W). By Theorem 26, $\mathbb{V}(\mathbf{H})$ is primitive.

Similarly, the second lattice in Figure 15 is $\mathbf{L}_{18}$, one of the lattices generating a join irreducible cover of $\mathbb{V}(\mathbf{L}_1)$. This time we note that $\mathbf{L}_{17} \leq \mathbf{L}_{18}[m]$, so by the lemma $\mathbf{L}_{18}$ is projective in any locally finite variety not containing $\mathbf{L}_{17}$. The other s.i. lattices in $\mathbb{V}(\mathbf{L}_{18})$ are $\mathbf{2}$, $\mathbf{N}_5$, and $\mathbf{L}_1$. Thus $\mathbb{V}(\mathbf{L}_{18})$ is primitive.

The same argument using $\mathbf{L}_{17}$ applies to the varieties generated by the lattices $\mathbf{L}_{24}$ and $\mathbf{L}_{25}$ that are covers of $\mathbb{V}(\mathbf{L}_1)$. (The figures of $\mathbf{L}_{20}$ and $\mathbf{L}_{21}$ in [14] are missing a coatom, making 2 more doubly reducibles.) Summarizing:

Theorem 27. *The lattices* $\mathbf{H}$, $\mathbf{L}_{18}$, $\mathbf{L}_{24}$, *and* $\mathbf{L}_{25}$ *each generate a primitive lattice variety.*

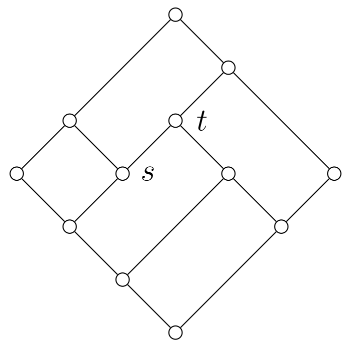


FIGURE 16. A lattice $\mathbf{F}$ with a proper (W)-failure that generates a primitive variety.

Corollary 28. *The varieties* $\mathbb{V}(\mathbf{L}_{17})$ *and* $\mathbb{V}(\mathbf{L}_{18})$ *are both primitive, but their join is not primitive.*

Proof. The lattice $\mathbf{L}_{17}$ is SIW, while $\mathbb{V}(\mathbf{L}_{18})$ is primitive by the preceding theorem. The lattice $\mathbf{L}_{18}[m]$ obtained by doubling the doubly reducible element m is a subdirect product of $\mathbf{L}_{17}$ and $\mathbf{L}_{18}$, and $\mathbf{L}_{18} \not\leq \mathbf{L}_{18}[m]$. Thus $\mathbf{L}_{18}$ is not weakly projective in the join of the varieties. By Theorem 3, the join is not primitive. □

5.2. **Second example.** Directly analogous to Theorem 26 is the following.

Theorem 29. *Let* $\mathbf{L}$ *be a finite s.i. lattice with a* (W)*-failure interval* I *satisfying the property of* Lemma 25(2). *If, in addition,*

(1) $\mathbf{L}[I] \notin \mathbb{V}(\mathbf{L})$, *and*
(2) *every s.i. lattice in* $\mathbb{HS}(\mathbf{L}) \setminus \{\mathbf{L}\}$ *is weakly projective in* $\mathbb{V}(\mathbf{L})$.

then $\mathbb{V}(\mathbf{L})$ *is a primitive lattice variety.*

The lattice $\mathbf{F}$ in Figure 16 contains the proper interval $I = [s, t]$ as its only (W)-failure. The other s.i. lattices in $\mathbb{V}(\mathbf{F})$ are $\mathbf{2}$, $\mathbf{N}_5$, $\mathbf{L}_{10}$, and $\mathbf{L}_{12}$, which satisfy (W). Thus $\mathbb{V}(\mathbf{F})$ is primitive.

The theorem also applies to $\mathbf{P}_5$ and the lattice $\mathbf{G}_{23}$ of Figure 20.

Theorem 30. *The lattices* $\mathbf{F}$, $\mathbf{P}_5$, *and* $\mathbf{G}_{23}$ *each generate a primitive lattice variety.*

Proof. It is straightforward to check that both $\mathbf{F}$ and $\mathbf{F}[I]$ are s.i. By the lemma, $\mathbf{F}$ is projective in the variety it generates. □

5.3. **Third example.** Now we turn to lattices with multiple doubly reducible elements, which turn out to be more difficult.

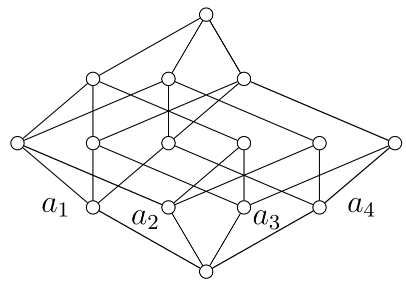


FIGURE 17. The lattice $\mathbf{U}_4$

The next (negative) observation limits the search for lattices that fail (W) but generate a primitive variety. Let $\mathbf{B}_n$ be the boolean lattice with n atoms. Consider the lattice $\mathbf{B}_n[c]$ obtained by doubling a coatom c. For $n \geq 2$, $\mathbf{B}_n[c]$ is s.i., and for $n \geq 4$ it fails (W), with many doubly reducible elements.

Lemma 31. *For $n \geq 4$, $\mathbf{B}_n[c]$ is not weakly projective in the variety it generates.*

Proof. We sketch the argument. Let c, d be coatoms of $\mathbf{B}_n$. Then $\mathbf{B}_n[c]$ has elements $(c, 0)$ and $(c, 1)$. Let $\mathbf{K}$ be the lattice obtained from $\mathbf{B}_n[c]$ by doubling the interval $[c \wedge d, (c, 0)]$. There is a homomorphism undoubling the interval, and another mapping $\mathbf{K}$ down to $[0, d] \cong \mathbf{B}_{n-1}[c \wedge d]$, so $\mathbf{K}$ is a subdirect product of $\mathbf{B}_n[c]$ and $\mathbf{B}_{n-1}[c \wedge d]$. However, $\mathbf{B}_n[c] \not\leq \mathbf{K}$ for $n \geq 4$. □

Now let us construct another lattice that fails (W) but generates a primitive variety. For an n-element set $X = \{a_1, \dots, a_n\}$ with $n \geq 3$, let $\mathbf{U}_n$ be the lattice of all subsets of X *except* $\{a_2, \dots, a_n\}$. Thus $\mathbf{U}_n$ is generated by its atoms subject to the relation $a_1 \leq a_2 \vee \dots \vee a_n$. That makes it a subdirectly irreducible, lower bounded lattice with critical quotient $[0, a_1]$. For example, $\mathbf{U}_3 \cong \mathbf{L}_1$, and $\mathbf{U}_4$ is drawn in Figure 17. (In the end, we can only use $\mathbf{U}_3$ and $\mathbf{U}_4$, but the general argument is sort of nice.)

Lemma 32. *For $n \geq 3$, the lattice $\mathbf{U}_n$ is projective in any locally finite variety excluding $\mathbf{L}_{15}$.*

Proof. Let $\mathbf{K}$ be a finite lattice excluding $\mathbf{L}_{15}$ as a sublattice, and let $f : \mathbf{K} \twoheadrightarrow \mathbf{U}_n$ be a surjective homomorphism. We want to find a retraction $g : \mathbf{U}_n \leq \mathbf{K}$.

Let z be the largest pre-image of 0_{U_n}. Then restriction $f' : \uparrow z \to \mathbf{U}_n$ is surjective, since $f(k \vee z) = f(k)$, and we have $(f')^{-1}(0_{U_n}) = \{z\}$. Then let w be the least pre-image of 1_{U_n} under f'. The restriction

$f'' : [z, w] \to \mathbf{U}_n$ is still surjective, and $(f'')^{-1}(1_{U_n}) = \{w\}$. To simplify notation, we may assume that $\mathbf{K} = [z, w]$ and $f = f''$.

Let $g_0 : \mathbf{U}_n \to \mathbf{K}$ be the map such that $g_0(u)$ is the largest pre-image of u for each $u \in U_n$. Then g_0 is meet-preserving and fg_0 is the identity on U_n. As usual, define

$$g_1(x) = g_0(x) \wedge \bigwedge_{\bigvee V \geq x} \bigvee_{v \in V} g_0(v).$$

Because $\mathbf{U}_n$ is a lower bounded lattice of rank 1, we can stop here; there is no need for the standard sequence g_n of maps as $g_n(x) = g_1(x)$ for $n \geq 1$. Thus $g = g_1$ is a join-preserving transversal. We want to show that, because $\mathbf{L}_{15} \not\leq \mathbf{K}$, it also preserves meets.

In fact, the map g is very easy to describe. Clearly $g(0_{U_n}) = z$, the only choice for a pre-image. Likewise $g(1_{U_n}) = w$. Also $g(a_1) = g_0(a_1)$: though a_1 has the nontrivial join cover $a_2 \vee \ldots \vee a_n$, that join is 1_{U_n}, which has a unique pre-mage w, which is the largest element of our modified $\mathbf{K}$. Hence $g(a_1) = g_0(a_1)$. The atoms a_j with $j \geq 2$ are join prime in $\mathbf{U}_n$, so $g(a_j) = g_0(a_j)$. The remaining elements of $\mathbf{U}_n$ are join reducible with a unique join decomposition. Therefore for $S \subseteq X$ we have $g(S) = \bigvee_{s \in S} g_0(s)$.

To simplify notation, let $b_j = g_0(a_j)$. Thus the image $g(U_n)$ consists of all joins of subsets of $Y = \{b_1, \ldots, b_n\}$ except $\{b_2 \vee \ldots \vee b_n\}$. We know from the construction that g preserves joins. This includes the fact that $g(a_2 \vee \ldots \vee a_n) = b_2 \vee \ldots \vee b_n = w = b_1 \vee b_2 \vee \ldots \vee b_n$.

Let E, $F \subseteq Y$, with neither one being $\{b_2, \ldots, b_n\}$. We will show by induction on $|E \cap F|$ that $\bigvee E \wedge \bigvee F = \bigvee (E \cap F)$. This means that meets are preserved, and g is a retraction.

If $E \cap F = \varnothing$, then $\bigvee E \wedge \bigvee F$ is a pre-image of 0_{U_n}, whence $\bigvee E \wedge \bigvee F = z$, which is $\bigvee \varnothing$ in our modified $\mathbf{K}$.

If $E \cap F = \{b_j\}$ say, then $b_j \in E$ implies $\bigvee E \geq b_j$, and similarly $\bigvee F \geq b_j$. Thus $\bigvee E \wedge \bigvee F \geq b_j$. Meanwhile $\bigvee E \wedge \bigvee F$ is a pre-image of a_j, and $b_j = g_0(a_j)$ was its largest pre-image. Therefore $\bigvee E \wedge \bigvee F \not> b_j$, and we get $\bigvee E \wedge \bigvee F = b_j$.

Now let $|E \cap F| = k \geq 2$, and assume that we have proved that meets are correct whenever the cardinality of the meet is less than k. Without loss of generality, E and F are incomparable.

Now $\mathbf{L}_{15}$ is a splitting lattice, whose splitting equation can be written as

$$a \geq b \ \& \ c \geq d \to (a \vee d) \wedge (b \vee c) \leq (a \wedge (b \vee c)) \vee (c \wedge (a \vee d)).$$

Choose B and D incomparable such that $B \cup D = E \cap F$. Then let $A = B \cup (E \setminus F)$ and $C = D \cup (F \setminus E)$. Then substitute the values

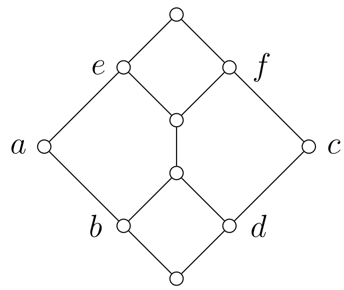


FIGURE 18. The lattice $\mathbf{L}_{15}$

Cardinality $n =$	5	6	7	8	9	10	11	12
Number of s.i. lattices	2	4	16	69	360	2103	13867	100853
s.i. lattices that fail (W)	0	0	0	4	55	629	6360	61634

TABLE 1. The number of s.i. lattices of cardinality ≤ 12 that fail (W).

$a = \bigvee A$, $b = \bigvee B$, $c = \bigvee C$, $d = \bigvee D$ into the splitting equation. We get $a \vee d = \bigvee E$ and $b \vee c = f = \bigvee F$, so that the left side becomes $\bigvee E \wedge \bigvee F$. The meets on the right side have cardinality less than k, so $a \wedge (b \vee c) = a \wedge f = \bigvee B$, and likewise $c \wedge (a \vee d) = \bigvee D$. Thus the right hand side is $\bigvee (E \cap F)$. Therefore the splitting equation (and induction) yields $\bigvee E \wedge \bigvee F = \bigvee (E \cap F)$, which was to be shown. $\square$

Now we look for the subdirectly irreducible lattices in $\mathbb{V}(\mathbf{U}_n)$, which include $\mathbf{2}$, $\mathbf{N}_5$, and $\mathbf{U}_j$ for $3 \leq j \leq n$. Unfortunately, it also includes $\mathbf{B}_{n-1}[c]$: take the sublattice generated by $a_2, \ldots, a_n$ and $\{a_1, a_2, \ldots, a_{n-1}\}$. For $k \geq 4$, $\mathbf{B}_k[c]$ is not weakly projective in the variety it generates. However, $\mathbf{U}_4$ contains only $\mathbf{B}_3[c] = \mathbf{L}_{14}$, besides the lattices listed above, so $\mathbb{V}(\mathbf{U}_4)$ is primitive.

Theorem 33. *The lattices $\mathbf{U}_3 \cong \mathbf{L}_1$ and $\mathbf{U}_4$ each generate a primitive lattice variety.*

6. Varieties generated by a s.i. with $|L| \leq 9$

Our working algorithm suffices to determine whether $\mathbb{V}(\mathbf{L})$ is primitive for all s.i. lattices with $|L| \leq 9$. The s.i. lattices of cardinality up to 12 were computed, and tested to see which ones satisfy (W) [13]. The number of s.i. lattices satisfying and failing (W) are given in Table 1.

If $|L| \leq 7$, then $\mathbf{L}$ satisfies (W).

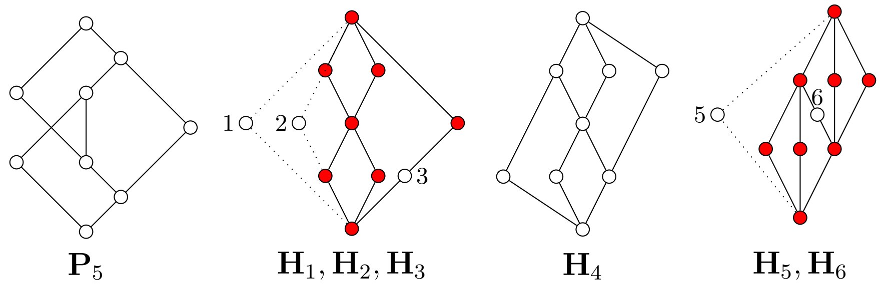


FIGURE 19. All 7 self-dual s.i. lattices with 9 elements that fail (W). For $\mathbf{H}_i$ $(i = 1, \ldots, 6)$ add element i, possibly connected by dotted lines.

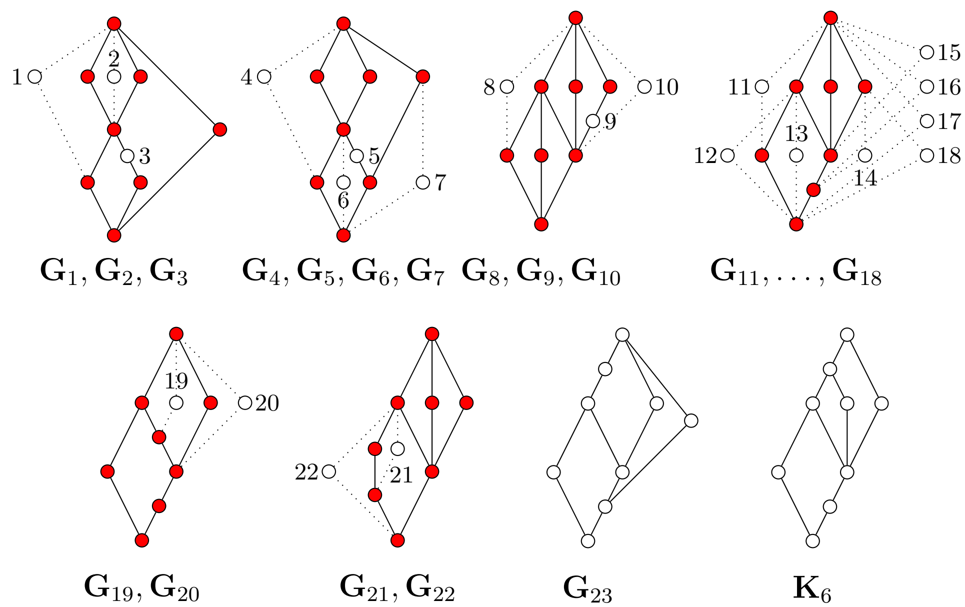


FIGURE 20. All 24 non-self-dual s.i. lattices with 9 elements that fail (W). For $\mathbf{G}_i$ $(i = 1, \ldots, 23)$ add element i, possibly connected by dotted lines. The dual lattices are $\mathbf{G}_1^d, \ldots, \mathbf{G}_{23}^d, \mathbf{K}_6^d$.

There are 69 s.i. lattices with 8 elements. Some 65 of these satisfy (W), hence also are IW and thus generate a primitive variety. The splitting lemma shows that $\mathbf{M}_{3,3}$, $\mathbf{V}_4$ and its dual $\mathbf{V}_5$ do NOT generate primitive varieties. On the other hand, since $\varpi(\mathbf{H}, \mathbf{H}[m])$ holds and $\mathbf{H}[m]$ is s.i., $\mathbb{V}(\mathbf{H})$ is primitive.

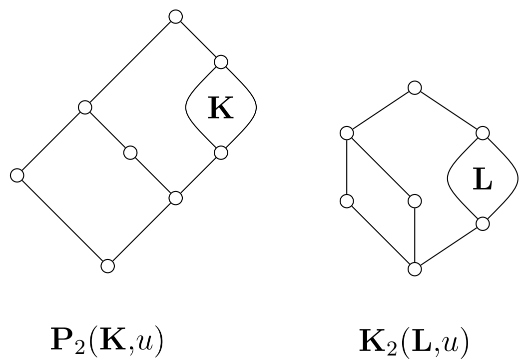


FIGURE 21. Two examples of the gluing construction that preserve the SIW property (class B) by Theorem 35.

There are 360 s.i. lattices with 9 elements. In fact, 304 of the 305 that satisfy (W) are IW, and hence generate a primitive variety. The exception is $\mathbf{H}[m]$, which does NOT generate a primitive variety.

Figures 19 and 20 give the 55 9-element s.i. lattices that fail (W). Some of those 55 that fail (W) contain $\mathbf{M}_{3,3}$, $\mathbf{V}_4$, or $\mathbf{V}_5$, and those automatically generate a non-primitive variety. However, 16 s.i. lattices with 9 elements DO generate a primitive variety: the self-dual lattices $\mathbf{P}_5$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, and the non-self-dual $\mathbf{G}_j$ $(1 \le j \le 5)$ and $\mathbf{G}_{23}$. Most (13) of the 16 have a unique doubly reducible element as their (W)-failure, but $\mathbf{P}_5$, $\mathbf{G}_{23}$ and its dual have a proper (W)-failure.

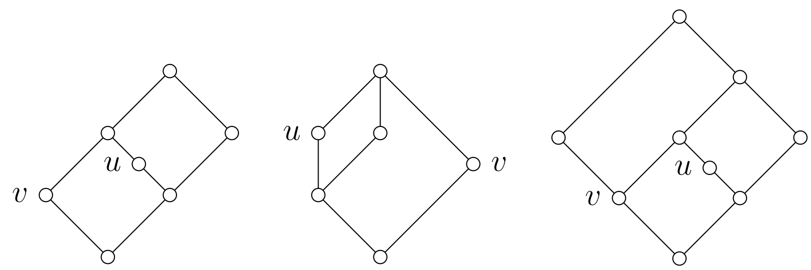
We have also shown that the larger lattices $\mathbf{F}$ and $\mathbf{U}_4$ generate primitive varieties.

7. Gluing constructions

The constructions of Section 3 can be extended. With the $N(\mathbf{K})$ construction as the model, we try the following plan. Take a finite s.i. lattice $\mathbf{L}$ and replace a critical interval $[u_*, u]$, where $u \in \mathrm{J}(\mathbf{L})$ and u_* is its lower cover, with a finite lattice $\mathbf{K}$. Denote this by $\mathbf{L}(\mathbf{K}, u)$. Figure 21 illustrates the gluing construction. We ask:

- When is $\mathbf{L}(\mathbf{K}, u)$ an IW or SIW lattice?
- When does $\mathbf{L}(\mathbf{K}, u)$ generate a primitive variety?

The answer depends on a number of factors, but there are at least 3 situations that produce positive results, which we denote as class (A), (B), and (C). Our examples are not intended to be exhaustive, just enough to see what is going on, using lattices from Figures 2 and 9.



$\mathbf{L}_3$ $\mathbf{L}_5$ $\mathbf{P}_4$

Figure 22. Some s.i. lattices with 2 non-isomorphic choices of a join irreducible critical quotient, $[u_*, u]$ and $[v_*, v]$.

For most of the examples in this section, there is an obvious, unique (up to isomorphism) choice of u for the critical quotient $[u_*, u]$. Of course, that need not be the case for an arbitrary finite s.i. lattice. Three exceptions are given in Figure 22, and dealt with in Theorem 34.

(A) The first possibility is that $\mathbf{L}(\mathbf{K}, u)$ can be obtained from the constructions C, N, U, D, Q, and Q^* of Section 3. In that event, if $\mathbf{K}$ is SIW then $\mathbf{L}(\mathbf{K}, u)$ is SIW. This applies when $\mathbf{L}$ is one of the lattices $\mathbf{M}_k$, $\mathbf{M}_k[a]$, $\mathbf{N}_5$, $\mathbf{L}_3$, $\mathbf{L}_4$, $\mathbf{L}_5$, $\mathbf{L}_6$, $\mathbf{L}_9$, $\mathbf{L}_{10}$, $\mathbf{L}_9^n$, $\mathbf{L}_{10}^n$, $\mathbf{P}_4$, $\mathbf{P}_{10}$, $\mathbf{K}_1$, with an appropriate choice of u. For example, $\mathbf{L}_3(\mathbf{K}, u) = U(N(\mathbf{K}), 0_K)$ and $\mathbf{L}_4(\mathbf{K}, u) = U(U(\mathbf{K}, 0_K), 0_K)$. Omitting the ones that also fall into class (B), we have this result.

Theorem 34. *Let* $\mathbf{L}$ *be one of the lattices* $\mathbf{M}_k$ $(k \geq 3)$, $\mathbf{L}_3$, $\mathbf{L}_4$, $\mathbf{L}_5$, $\mathbf{P}_4$, *or* $\mathbf{P}_{10}$, *and let* $[u_*, u]$ *be a critical quotient with* $u \in \mathrm{J}(\mathbf{L})$. *If the lattice* $\mathbf{K}$ *is SIW, then* $\mathbf{L}(\mathbf{K}, u)$ *is SIW.*

Proof. For each of these lattices, with one choice of u, the gluing can be obtained from the constructions of Section 3. For example, this applies to the lattices $\mathbf{L}_3$, $\mathbf{L}_5$, or $\mathbf{P}_4$ with u as labeled in Figure 22. With the exception of $\mathbf{M}_k$ and $\mathbf{L}_4$, there is a non-isomorphic choice v of the join irreducible critical quotient. But then, with these small lattices, *ad hoc* arguments can be used to show that the SIW property is preserved by gluing at $[v_*, v]$. □

(B) For the second case, we consider when $\mathbf{L}$ is s.i., and the critical interval $[u_*, u]$ is a pair of doubly irreducible elements. This is a direct generalization of $N(\mathbf{K})$, but now we do not know exactly the lattices in $\mathbb{HS}(\mathbf{L})$, so more care is required. Nonetheless, the SIW property is preserved in this case.

Theorem 35. *Let* $\mathbf{L}$ *be a finite, SIW lattice with* $u \in \mathrm{J}(\mathbf{L})$ *such that both* u *and its lower cover* u_* *are doubly irreducible. Assume further that* $\uparrow u$ *and* $\downarrow u_*$ *are chains in* $\mathbf{L}$. *If* $\mathbf{K}$ *is a finite SIW lattice, then* $\mathbf{L}(\mathbf{K}, u)$ *is also SIW, and hence* $\mathbb{V}(\mathbf{L}(\mathbf{K}, u))$ *is primitive.*

Proof. First consider sublattices. Think of $\mathbf{L}(\mathbf{K}, u)$ as $L \cup K$ with the identifications $u \equiv 1_K$ and $u_* \equiv 0_K$, and the transitive closure of the inherited order. Then a sublattice $\mathbf{P} \leq \mathbf{L}(\mathbf{K}, u)$ is of the form $P = T \cup S$ with $\mathbf{T} \leq \mathbf{L}$ and $\mathbf{S} \leq \mathbf{K}$. Let us show that $\mathbf{P}$ satisfies (W).

Offhand, we do not know whether u or u_* are in T or S, but 1_S and 0_S play the corresponding roles.

- $u_* \leq 0_S \leq 1_S \leq u$,
- 0_S is join irreducible in $\mathbf{P}$, and 1_S is meet irreducible,
- if $t \in T \setminus S$ and $t \nleq 0_S$, then $t \vee 0_S \geq 1_S$.

Assume $a \wedge b \leq c \vee d$ in $\mathbf{P}$. Since $\mathbf{T}$ satisfies (W), we may assume that at least one of these elements is in S. By duality, say $a \in S$. If $c \vee d \geq 1_S$, the conclusion of (W) holds, so assume $c \vee d \ngeq 1_S$. Without loss of generality a and b are incomparable, c and d are incomparable. Combining the assumptions $c \parallel d$ and $c \vee d \ngeq 1_S$, either both c, d are in S, or neither is.

If a, c, $d \in S$, then either $b \in S$ and we can apply (W) in $\mathbf{S}$, or $a \wedge b \leq 0_S \leq c$ since $a \parallel b$. If c, $d \notin S$, then $b \notin S$, using the fact that 0_S is join irreducible. But then, using $a \parallel b$, we have $a \wedge b = u \wedge b \leq c \vee d$ in $\mathbf{L}$. Applying (W) yields the desired conclusion, since $a \leq u$.

Now let θ be a congruence on $\mathbf{P}$. Note that the restrictions $\theta|_T$ and $\theta|_S$ are congruences on $\mathbf{T}$ and $\mathbf{S}$, respectively. By hypothesis, $\mathbf{T}/\theta$ and $\mathbf{S}/\theta$ satisfy (W). Without loss of generality, $\theta|_T$ and $\theta|_S$ collapse neither part entirely.

Of course, it is possible that $(t, s) \in \theta$ for some pairs $t \in T$, $s \in S$. (Otherwise the argument is exactly as before.) A crucial observation is that *if* $(t, t') \in \theta \cap T^2$ *and* $t \geq 1_S$, *then* $t' \geq 1_S$. For $(t, t') \in \theta$ implies $(t \wedge 1_S, t' \wedge 1_S) = (1_S, t' \wedge 1_S) \in \theta$. If $t' \ngeq 1_S$, then $t' \wedge 1_S \leq 0_S$, and $(1_S, t' \wedge 1_S) \in \theta$ means that θ collapses all of S, contrary to assumption. In terms of congruence classes, this can be written as $[1_S] \leq [t]$ implies $1_S \leq t$. Dually, $[0_S] \geq [t]$ implies $0_S \geq t$.

Moreover, the assumption that $\uparrow u$ is a chain in $\mathbf{L}$ means that $[1_S]$ is meet irreducible in $\mathbf{P}/\theta$, and similarly $\downarrow u_*$ being a chain implies that $[0_S]$ is join irreducible.

With these two observations, the preceding arguments can be adapted for an inclusion $[a] \wedge [b] \leq [c] \vee [d]$ in $\mathbf{P}/\theta$. The details are left to the reader. □

Theorem 35 applies, and gives a new construction, when $\mathbf{L}$ is one of the following lattices or their duals: $\mathbf{N}_5$, $\mathbf{L}_6$ or more generally $\mathbf{L}_6^k = N^k(\mathbf{N}_5)$, $\mathbf{L}_9$, $\mathbf{L}_9^k$, $\mathbf{L}_{16}$, $\mathbf{P}_2$, $\mathbf{P}_6$, $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_5$, $\mathbf{V}_1$ or more generally $\mathbf{M}_k[a]$, $\mathbf{V}_2$. Two of these are illustrated in Figure 21. The lattice $\mathbf{K}$ can be any SIW lattice.

(C) It may happen that $\mathbf{L}(\mathbf{K}, u)$ contains a single doubly reducible element. For example, if the critical join irreducible u of $\mathbf{L}$ is meet reducible, and 1_K is join reducible, then $u = 1_K$ will be doubly reducible in $\mathbf{L}(\mathbf{K}, u)$. In some instances we can apply Theorem 26 to show that $\mathbb{V}(\mathbf{L}(\mathbf{K}, u))$ is primitive. This would require that $\mathbf{L}(\mathbf{K}, u)[u]$ not be in $\mathbb{V}(\mathbf{L}(\mathbf{K}, u))$, and more (see below). It seems to be rare, but there are at least a few cases where the gluing works to generate a primitive variety.

First consider $\mathbf{L}_1(\mathbf{K}, u)$, as diagrammed in Figure 23. We need to avoid having the lattice $\mathbf{Y}$ of that figure as a sublattice, since it does not generate a primitive variety by Corollary 5. That restricts the options for $\mathbf{K}$: the join of any 2 distinct atoms must be 1_K.

Lemma 36. *If* $\mathbf{K}$ *is one of* **2**, **3**, $\mathbf{M}_k$ $(k \geq 2)$, $\mathbf{N}_5$ *or more generally* $\mathbf{M}_k[a]$, *then* $\mathbb{V}(\mathbf{L}_1(\mathbf{K}, u))$ *is primitive.*

Proof. Let us consider the case where $\mathbf{K} = \mathbf{M}[a]$, since that contains the other options. The s.i. lattices in $\mathcal{V} = \mathbb{V}(\mathbf{L}_1(\mathbf{M}_k[a], u))$ are $\mathbf{L}_1(\mathbf{M}_\ell[a], u)$ for $2 \leq \ell \leq k$, $\mathbf{L}_1(\mathbf{M}_\ell, u)$ for $2 \leq \ell \leq k$, $\mathbf{M}_\ell[a]$ for $3 \leq \ell \leq k$, $\mathbf{M}_\ell$ for $3 \leq \ell \leq k$, $\mathbf{L}_{17}$, $\mathbf{L}_1$, $\mathbf{N}_5$, and **2**. Many of these lattices satisfy (W), and we can apply Lemma 25(1) to show that the rest are projective in $\mathbb{V}(\mathbf{L}_1(\mathbf{M}_k[a], u))$ once we observe that $\mathbf{L}_1(\mathbf{M}_2, u)[u]$, which is the lattice $\mathbf{Y}$ in Figure 23, is not in the variety. But $\mathbf{Y}$ satisfies (W), so it is projective in the class of finite lattices. Since $\mathbf{Y} \not\leq \mathbf{S}$ for any s.i. in the above list, so $\mathbf{Y} \notin \mathcal{V}$. We conclude by Theorem 26 that $\mathcal{V}$ is primitive. □

By a directly analogous argument, with the lattice $\mathbf{K}_3$ in place of $\mathbf{Y}$, we obtain another case.

Lemma 37. *If* $\mathbf{K}$ *is one of* **2**, **3**, $\mathbf{M}_k$ $(k \geq 2)$, $\mathbf{N}_5$ *or more generally* $\mathbf{M}_k[a]$, *then* $\mathbb{V}(\mathbf{L}_{19}(\mathbf{K}, u))$ *is primitive.*

Likewise, there is the construction where a (non-critical) interval $[0, a]$ of $\mathbf{L}_5$ is replaced by $\mathbf{K}$ to obtain $\mathbf{L}_5(\mathbf{K}, a)$; see Figure 24. The same considerations apply to this construction, with the lattice $\mathbf{H}[m]$ of Figure 4 playing the role of $\mathbf{Y}$.

Lemma 38. *If* $\mathbf{K}$ *is one of* **2**, **3**, $\mathbf{M}_k$ $(k \geq 2)$, $\mathbf{N}_5$ *or more generally* $\mathbf{M}_k[a]$, *then* $\mathbb{V}(\mathbf{L}_5(\mathbf{K}, a))$ *is primitive.*

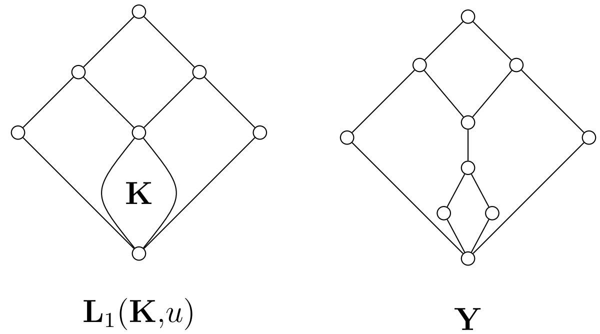


FIGURE 23. The $\mathbf{L}_1(\mathbf{K}, u)$ construction (class C) and the lattice $\mathbf{Y}$ used in the proof of Lemma 36.

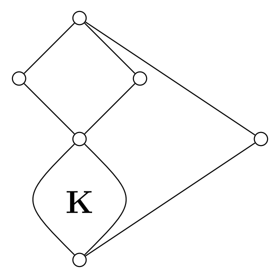


FIGURE 24. The $\mathbf{L}_5(\mathbf{K}, a)$ construction

8. Questions

In this paper, we have studied primitive lattice varieties, that is, varieties such that every subquasivariety is a subvariety. Every locally finite, inherently Whitman lattice variety is primitive, and we have shown that there are uncountably many of that type. But we have also found primitive lattice varieties $\mathbb{V}(\mathbf{L})$ where $\mathbf{L}$ fails Whitman's condition (W).

Some interesting problems remain.

(1) We conjecture that for a s.i. lattice $\mathbf{K}$ in a locally finite variety $\mathcal{V}$,
  - $\mathbf{K}$ is weakly projective in $\mathcal{V}$ iff $\mathbf{K}$ is projective in $\mathcal{V}$,
  - $\mathbf{K}$ is projective in $\mathcal{V}$ iff $\mathbf{K} \leq \mathbf{F}_{\mathcal{V}}(X)$ for some X.

  The conjectures are true for the case when $\mathcal{V} = \mathbb{V}(\mathbf{K})$ by Theorem 9. One way to look at the more general situation is as a finitely presented lattice problem (relative to $\mathcal{V}$). Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(X)$ and let $f : \mathbf{F} \twoheadrightarrow \mathbf{K}$ be a surjective homomorphism. Consider the set $\mathfrak{T}$ of all sublattices $\mathrm{Sg}_{\mathbf{F}}(T)$ generated by transversals of f, i.e., $T = g(K)$ for some $g : K \to F$ such that $fg = \mathrm{id}_K$, but g need not be a homomorphism. *What are the minimal members of $\mathfrak{T}$ with respect to sublattice inclusion?* For $\mathbf{K}$ is projective in $\mathcal{V}$ iff $\mathbf{K}$ is the unique (up to isomorphism) minimal member of $\mathfrak{T}$. The mappings g_k allow us to assume some partial lattice structure on T.

(2) Theorem 3 shows that it is decidable whether a finite lattice generates a primitive variety. Is there a reasonably efficient algorithm? Is there a reasonable algorithm at least to decide whether $\mathbb{V}(\mathbf{L})$ is primitive when $\mathbf{L}$ satisfies (W)?

(3) Example 19 shows that the lattice $\mathbf{C}_1$ from Ježek and Slavík's list of s.i. sublattices of a free lattice [12] generates a variety that is not primitive. What about the remaining lattices in their list? We know that $\mathbf{A}_2$, $\mathbf{A}_4$, and $\mathbf{B}_n$ are not SIW, but suspect that most of their lattices are IW.

(4) If $\mathbb{V}(\mathbf{L})$ fails to be primitive, is it Q-universal as a quasivariety?

(5) Our gluing constructions are rather *ad hoc*. Is there a more systematic explanation of which gluings can be used?

(6) Is there a non-primitive lattice variety $\mathcal{V}$ such that $\mathbf{L}[I] \in \mathcal{V}$ for all s.i. $\mathbf{L} \in \mathcal{V}$ and (W)-failure intervals I in $\mathbf{L}$? This would say that in Theorem 3, you must choose a different witness $\mathbf{L}'$ of non-primitivity besides some $\mathbf{L}[I]$. There is also the generalized doubling construction of Day [5] to consider, which applies to overlapping (W)-failures.

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