HOW ALIENS DO MATH

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Part I: General Considerations

1. Pioneers and Voyagers

Attempts to communicate with the inhabitants of other solar systems have been made since 1959. These initially involved sending and listening for radio transmissions. In the 1970's, messages to aliens were included on the Pioneer 10 and 11, and Voyager 1 and 2 spacecraft. There is an account of these efforts in the book [19]; see especially the chapter by F. D. Drake. The Voyager messages include some mathematics, e.g., a translation table between binary and decimal representation systems. (Recent updates on the search for extra-terrestial intelligence may be found at *www.seti-inst.edu* and *www.setileague.org*.)

At a NATO Workshop at the University of Hawaii in October 2002, Jack Cohen raised the question: *What makes you think that the aliens will recognize your mathematics? Might they not have an entirely different mode of thought?* At the time, the feeling of the mathematicians present was that the fundamentals of discrete math, and in particular the integers, would be the same in any mathematical system. While aliens might have a different perception that changes the more complex structures that they would use to describe the universe, their natural numbers would be the same as ours. Furthermore, if they are very intelligent, then they should be able to add and multiply integers, so for example aliens would recognize a sequence of prime numbers.

Now in fact, such questions about the fundamentals of mathematics have a long history, without the alien factor. My own interest was spurred years ago by a rather long but otherwise forgettable lecture on "The nature of duality" by a visitor to the Philosophy Department. But Jack's question has encouraged me to put down some of the thoughts that have been fomenting beneath the surface, in need for further development. Besides raising a number of topics for discussion, we will propose some problems that might be appropriate for student projects.

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Thanks to my friends on earth, and also colleagues and students at the University of Hawaii, for many helpful discussions and comments. Contributors to the discussion include David Casperson, Micah Chrisman, Jaroslav Ježek, Jon Kalk, Bill Lampe, Leonid Levkovitch-Maslyuk, Adolf Mader, Brandon Muranaka, Dale Myers, Glenn Rediger, G. W. Ross, Victor Sergeev, and George Wilkens.

Let us begin with an obvious question. In dealing with aliens who don't understand our mathematics, there seem to be at least three possibilities.

(1) Maybe they aren't very smart, like worms or horses.
(2) Maybe they are intelligent, but don't have a culture of mathematics, like people a few thousand years ago.
(3) Maybe they are intelligent, but have a totally different mode of perception, which does not necessarily include discreteness.

How can we distinguish between these possibilities?

2. Kronecker

Recall Kronecker's famous remark that *"God created the natural numbers; all the rest is the work of man."* There is a lot to be said for this position. A fundamental question is: Are the natural numbers more basic than set theory? In terms of interpretability, of course, set theory is more general (see, e.g., Roitman [17], Chapter 4). But historically, set theory makes a rather late appearance on the scene, and its abstract formalization is still not well-understood by most professional mathematicians.

For the purposes of discussion, it is useful to delineate some different parts of mathematics:

(1) counting,
(2) logic,
(3) set theory,
(4) discrete structures (algebra),
(5) continuous structures (geometry, analysis, topology).

The problem is to understand to what extent each of these parts has a Platonic reality, and to what extent they depend upon our perception and mental construction/interpretation of our surroundings?

Here we will primarily deal with the ontology of discrete (even finite) systems. The second major issue, for another time, is the nature of the continuum. References there include the work of Dedekind, Cantor, C. S. Peirce, Weyl, Brouwer, Heyting, Bishop, A. Robinson and J. L. Bell. It seems clear that our modeling of the continuum is more arbitrary than the system of natural numbers.

It is reasonable to ask to what extent an alien's logic would resemble ours. In Part II, we will explore some alternative forms of logic.

3. Dinosaurs and Plato

Imagine two dinosaurs walking through the forest. What does that mean? With no men there to count them, in what sense are there really two dinosaurs? Are numbers a product of our minds, or do they have some sort of Platonic independent existence?

We can ask similar questions about the real numbers or the Continuum Hypothesis, but this is more basic. Kronecker's distinction may indeed have some validity. Surely we would want to allow those two dinosaurs to be there, whether or not they leave two fossil skeletons for us to count later.

A related question is: What are we to make of the role of number-theoretic properties in nature? For example, why are there 13- and 17-year periodic cicadas? Where else do prime numbers occur in nature? What is the significance of Fibonacci numbers in nature? It would seem at the very least that the answers to these questions do not depend on our perception of the phenomena.

Now imagine a rainbow over the two dinosaurs walking through the forest. What does *that* mean? In a sense, no two people ever see the same rainbow. Nor do we have any way of knowing how colors actually appear to other people. Our analysis of *color* should be quite different from that of *number*, the former being much more dependent on our perception than the latter.

4. Cardinality

More generally, what is *cardinality?* We know that the predicate that two sets *have the same cardinality* depends heavily on our choice of set theory, with the obvious examples being countable models of set theory and the independence of the Continuum Hypothesis. Moreover, even though equivalence relations on classes are logically permissible, they are troubling as a basic idea, even for finite cardinals.

How is 2 different from 10^{75}? from $\aleph_0$? from an inaccessible cardinal? from a Dedekind-finite infinite cardinal?

It seems a good guess that primitive life forms, say worms, cannot count. They would have no clear concept of *two.* On the other hand, more advanced species such as dogs or cats may well recognize a form of duality. Animals such as crows, dolphins or chimpanzees perhaps can count higher. What numbers does a human baby recognize? These matters are discussed at length in Dehaene [4], and summarized in Devlin [6].

In most (but not all) numeral systems, the numerals for one to three consist of that many dots or lines, perhaps in a cursive or stylized form. This is often true of four, but for five onwards the symbols become more abstract. See Cajori [3] and Ifrah [10].

Well into the twentieth century, there were societies in Africa, Oceania and America where people counted *one, two, three, four, many.* Here, it is clearly not a matter of intelligence, but of culture. Indeed, one imagines that all of mankind must have been like that in the recent past. For larger counts, such devices as notched sticks or knotted ropes are used, so there is a concept of cardinality beyond the naming of numbers. See the first chapter of Ifrah [10].

5. Theories with No Discrete Model

It is easy to construct theories with no discrete models, e.g., the theory of dense chains.

(1) $\forall x \quad x \leq x$
(2) $\forall x \forall y \quad x \leq y \ \& \ y \leq x \implies x = y$
(3) $\forall x \forall y \forall z \quad x \leq y \ \& \ y \leq z \implies x \leq z$
(4) $\forall x \forall y \quad x \leq y$ OR $y \leq x$
(5) $\forall x \forall y \quad x \leq y \ \& \ x \neq y \implies \exists z \quad x \leq z \ \& \ z \leq y \ \& \ x \neq z \ \& \ z \neq y$

And we can imagine creatures for whom dense chains are as fundamental to their way of thought as discrete sets are to us. But we also know that ordered sets can be represented by set inclusion, that is, can be embedded in a direct power of **2** (the two-element ordered set). So, at least in a sense, if these creatures recognize dense chains but not **2**, then it represents ignorance on their part.

This leads us to an important observation made by George Wilkens:

> **Wilkens' Thesis:** If an intelligent being can understand order, then it can comprehend the discrete set **2**.

The idea is that if a partially ordered system is included in its mode of thought, then a mathematically inclined alien should also be able to take reducts to have just an ordered set, and then take ideals (or Dedekind cuts for a total order). It suffices to take principal ideals, which requires only a minimal version of set theory. Thus the mathematics of an intelligent creature, but one that does not comprehend discrete systems, should be based on unordered structures, or else on cyclic structures, such as the circle group (complex numbers with $|z| = 1$ under multiplication), or better yet the unit circle with the operation $u * v = uv^{-1}$.

6. Set Theory with No Discrete Model

Our alien friends should have some means for dealing with collections of objects. We should not expect that it would be formalized exactly by the axioms of ZFC, but we can ask whether in its naive form it would not resemble our naive set theory. One approach to this question is to consider modifications to standard set theory.

(1) Can we have a set theory that has models not containing a 2-element set?
(2) Can we have a set theory that has no model containing a 2-element set?
(3) Can we have a set theory that contains a 2-element set but not an n-element set for every n? (For comparison, it is easy to have an n-element set for every integer $n \geq 0$, but no infinite set.)
(4) Does it make sense to have the logic contain **2** but not the set theory?

In this context, it is not even clear what we mean by standard set theory. There are variations of the axioms for ZFC, and they need not be equivalent once you start dropping or weakening their parts. Two standard choices for amateurs are Devlin [5] and Roitman [17].

It is possible to weaken set theory until there are models in which no nonempty set is finite, and then add an axiom that every nonempty set must be infinite. In particular, let us pose the problem of developing a version of set theory that would satisfy the following axiom:

$$\forall x\ x \neq \emptyset \implies \exists b\ b \in x\ \&\ x - \{b\} \neq \emptyset\,.$$

Of course, some of the original axioms will have to be modified significantly. The question is whether this can be done in such a way that anything recognizable is left on the skeleton.

Dale Myers points out that in Boolean-valued set theory, or fuzzy set theory, every nonempty set has a nonempty proper subset. However, in these cases the meaning of *membership* is changed: the truth value of the relation $\in (b, x)$ is taken to be an element in an atomless Boolean algebra, or the real interval $[0, 1]$, respectively.

Of course, it is easy to remove the Axiom of Infinity and have a set theory which has a model with only finite sets.

7. Addition and Multiplication

Let us consider the arithmetic of the natural numbers. Suppose we encounter an alien whose mode of thought gives him access to an unlimited collection of discrete, finite sets.

First of all, can he abstract the notion of cardinality, that is, does he think of numbers *per se*? Can he fill in the gaps in the sizes of the collections he has, and extend beyond them? In other words: Can he count? We don't necessarily require him to think of the infinite collection $\mathbf{N}$, though it is convenient for us to speak that way.

Next, we would expect that an intelligent being who could count, could also put the natural order on the natural numbers.

The addition operation can be viewed in several ways: using the successor function, disjoint union, combinations of lengths (archaic but legitimate), and perhaps others. Our creature's logic would presumably support this operation, or some variant. One viable alternative would be a creature that does only modular arithmetic.

Multiplication is harder. It could be thought of as successive addition, direct products, areas, or other ways. Again, we would expect the logic to support some version of this operation. It is worthwhile to appreciate the great step taken by Descartes in *La Géométrie*, wherein he views a product as a number (scalar), rather than an area.

Beyond this are all sorts of complications: zero, subtraction, negative numbers, rational numbers, real numbers, imaginary and complex numbers. We should consider the historical difficulties which mankind has had with each of these concepts, and also the notational difficulties and how this influences thought. Multiplying Roman numerals is difficult: what is XLII $\times$ MCXXIV? It would be mere coincidence if our aliens used a decimal system - if, indeed, they use a place system at all. What are the alternatives for numerals?

What if our alien knew how to do finite fields - and only those?

8. Conclusions

Here are some fundamental questions.

(1) Is set theory a preferable place to lay the foundations of mathematics, as opposed to the natural numbers (or some third alternative)? Since **N** can be interpreted in set theory, then the latter is more general. On the other hand, numbers definitely precede sets historically, and are more intuitive. Also, set theory contains an inescapable indefiniteness, especially with regard to the Power Set Axiom. (The richness of functions available in a model of set theory depends upon the richness of subsets.) Thus there are different models of set theory, but only one (up to isomorphism) for the natural numbers. Kronecker would definitely vote in favor of **N** as a basis for mathematics. Constructivists would go further, and suggest that we need to start with a different logic.

(2) How do our perception, and the workings of our brain, figure into the picture? This question has been considered from Kant onwards, with more modern references including Devlin [6] and Lakoff and Núñez [18]. For a simple example: How do blind mathematicians perceive continuity? (See Jackson [11].) Does sight influence our concept of discreteness? There are at least three possibilities to consider. (i) What if our senses of sight, hearing, and touch were absent or undeveloped, as our senses of taste and smell are relatively undeveloped? (ii) What if the world we inhabited actually contained no distinct objects, that is, if reality were blurry on the scale of our perception? Consider the worm. (iii) Or what if we had additional senses, hard for us to conceive now?

(3) Can we imagine a "world form" with no analogue of the discrete natural numbers? For example, the interstellar messages sent from earth presuppose that the aliens will understand the binary system.

(4) Can we expect our putative intelligent aliens to have formalized their mathematics? Apparently, for most of their history, humans did not have much mathematics beyond rudimentary counting. Nonetheless, axiomatization appears at a relatively early date with the ancient Greeks. Compare different approaches to abstraction culturally.

(5) If we have the natural numbers **N** and the aliens don't, then isn't that a reflection of their ignorance? Can we say the same thing about set theory? What

features should a system of thought have have in order to be considered *mathematics?* Certainly we should expect our aliens to have a consistent system, but not all consistent systems are equally powerful (or even comparable).

(6) Perhaps they have something that we don't. *Could* aliens have something inexplicable in our system of math and logic? There is a principle of relativity here: the question is not *How do aliens do math?* but rather *How does one mathematical system interpret into another?* It goes both ways.

(7) The analogy with language is perhaps apropos. Language influences thought. Humans communicate in many different languages, all reasonably powerful, and it possible to translate from one language to another. However, there is generally some meaning lost in translation, because the language itself builds in nuances of interpretation. This expressive power of language affects the direction of thought, and the same must be true of the language of mathematics.

Part II: More Technical Considerations

In this section we will describe some types of generalized logic, which have the same general form as our various logics, but perhaps different structures and interpretations. Our guiding theme will be to treat logic as algebra, following a well-established (and profitable) tradition; [1], [9] and [12] are good introductions. We will phrase the discussion in the language of universal algebra, to make it broad enough to include not only traditional logics, but also fuzzy logic and the logic of quantum computing; see [15], [16], [20], [8], [13] and [14] for introductions to these. For the sake of simplicity we avoid many extra features that could legitimately be considered, such as partial algebras, multi-algebras, non-finitary types, relations of non-specified arity (rank), and time dependence.

We should observe a distinction, that we are describing formalizations of mathematics, not mathematical thought as it actually occurs in the brain. The two are conceivably quite different.

A *logical system* has the following elements.

0. A *prelogic* consisting of those mathematical terms that we use to describe a system from the outside.

(i) There should be enough (naive) set theory to describe a set of variables, functions and relations.

(ii) There should be some (primitive) notion of equality.

(iii) We require the notion of an algebra or relational structure.

(iv) Some specific algebras or relational structures are assumed to be known for any particular mode of thought.

(v) We require enough recursion to define term algebras and homomorphisms therefrom.

These prelogical needs should be more strictly surveyed at some point.

As an example, standard propositional logic presupposes notions equivalent to a 2-element Boolean algebra, and the ability to describe Boolean terms and evaluate Boolean expressions. However, it does not initially require the concept of an arbitrary Boolean algebra.

1. An *alphabet* containing
   - X a set of variable symbols,
   - C constant symbols,
   - F function symbols,
   - R relation symbols.

The alphabet also contains the symbols (,) for parentheses and commas, and = for equality.

The *language* which we employ consists of the *alphabet* for symbols, the *type* to establish syntax, and the notion of *interpretation* to determine semantics. The latter two will be described below.

(Since constants may be regarded as nullary functions, it is not necessary to list them separately from the proper function symbols. However, we follow the archaic practice of doing so for the sake of clarity.

For practical purposes, we want to assume that X and R are nonempty, i.e., that there are some variables and some relation symbols. Otherwise, some of what follows is vacuous. There are situations where one wants to treat all the elements of a structure as constants, and consider sentences involving those constants only, with no variables and no quantifiers. These cases allow some simplification.)

2. A *type* $\tau = \langle C_\tau, F_\tau, R_\tau, \alpha_\tau, \mathbf{A}_\tau \rangle$ where
   (i) $C_\tau \subseteq C$,
   (ii) $F_\tau \subseteq F$,
   (iii) $R_\tau \subseteq R$,
   (iv) $\alpha_\tau : F_\tau \cup R_\tau \to \mathbf{N}^+$,
   (v) $\mathbf{A}_\tau$ is a known structure (see below).

The interpretation is that C_τ is the set of names for constants in τ-structures, F_τ is the set of proper (non-nullary) function symbols, and R_τ is the set of relational symbols. The mapping α_τ into the positive integers gives the arities (ranks) of the functions and relations of τ. The relations on τ-structures take their values in $\mathbf{A}_\tau$, which has a different type λ. This process of types referring to types must end in finitely many steps at a type with no relational symbols. We adopt the convention that, for a type μ, $\mathbf{A}_\mu = \emptyset$ if and only if $R_\mu = \emptyset$. (The scheme under construction, in its most general version, is illustrated in Figure 1 with $\tau = \tau_0$ and $\lambda = \tau_1$. Normally the syntax structure would have only 2 or 3 levels ($n = 1$ or 2), but one can make interesting examples with more.)

Here τ represents the type of the structures that we are describing, whereas λ is the type of the logic. Since we want to form predicates in τ-structures, we assume that R_τ is nonempty. If we want to talk about *equality* in τ-structures, then there should be a relational symbol for that in R_τ.

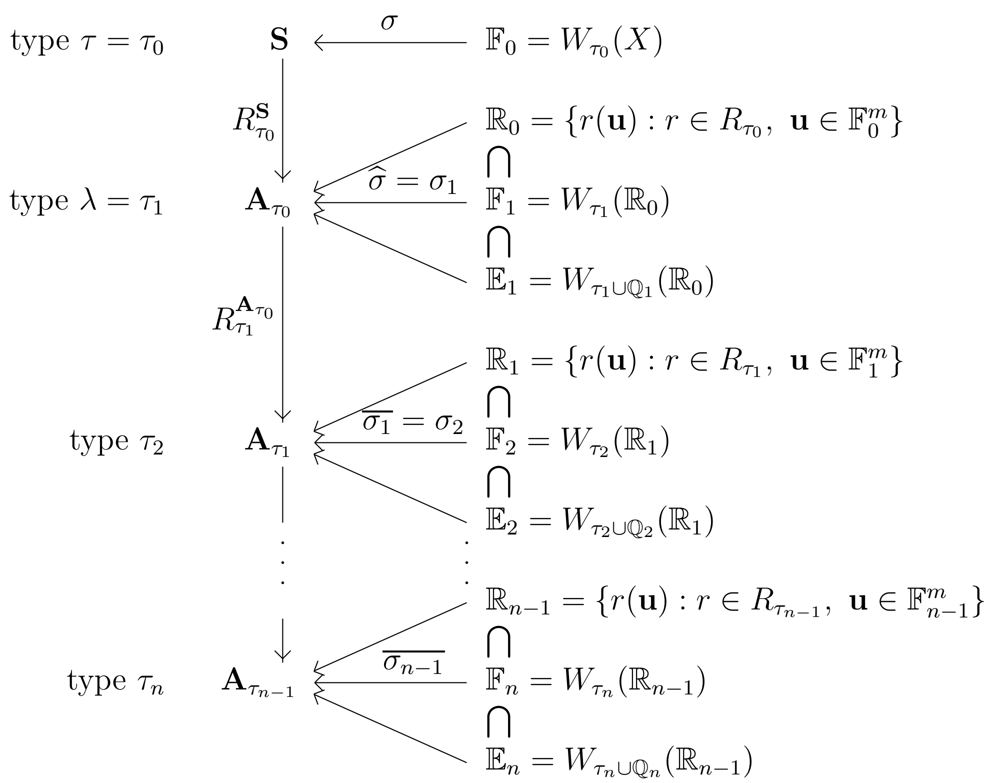

FIGURE 1

3. A *structure of type* τ is a system $\mathbf{S} = \langle S; C^{\mathbf{S}}_\tau, F^{\mathbf{S}}_\tau, R^{\mathbf{S}}_\tau \rangle$ where
   (i) S is the carrier set of $\mathbf{S}$,
   (ii) for each $c \in C_\tau$, $c^{\mathbf{S}}$ is a constant (nullary operation) of $\mathbf{S}$,
   (iii) for each $f \in F_\tau$, $f^{\mathbf{S}} : S^{\alpha_\tau(f)} \to S$,
   (iv) for each $r \in R_\tau$, $r^{\mathbf{S}} : S^{\alpha_\tau(r)} \to \mathbf{A}_\tau$.

Thus $c^{\mathbf{S}}$, $f^{\mathbf{S}}$ and $r^{\mathbf{S}}$ denote the realization in the concrete structure $\mathbf{S}$ of the corresponding symbols in C_τ, F_τ and R_τ, respectively. In our extended definition of a relational structure, the relations take their values in $\mathbf{A}_\tau$. To recover the standard case, where $r^{\mathbf{S}}$ is a subset of $S^{\alpha_\tau(r)}$, we take $\mathbf{A}_\tau$ to be the Boolean algebra $\mathbf{2}$, and identify a relation $r^{\mathbf{S}}$ with its characteristic function.

4. The algebra $\mathbf{A}_\tau$ is the structure in which our *logic* takes its truth values, i.e., the truth value of an elementary proposition ($\mathbb{F}_1$ below) will be an element of $\mathbf{A}_\tau$. It should be assumed to be a known, concrete structure. $\mathbf{A}_\tau$ will have a presumably different type $\lambda = \langle C_\lambda, F_\lambda, R_\lambda, \alpha_\lambda, \mathbf{A}_\lambda \rangle$. Again, if $\mathbf{A}_\tau$ is an algebra with no relations, i.e., if $R_\lambda = \emptyset$, then $\mathbf{A}_\lambda$ is just the empty set.

5. Form the term algebra $\mathbb{F}_0 = W_\tau(X)$, i.e., the free closure of $X \cup C_\tau$ under the function symbols of τ. Explicitly,

(i) $X \cup C_\tau \subseteq \mathbb{F}_0$;
(ii) if $f \in F_\tau$ has arity m and $\mathbf{t} \in \mathbb{F}_0^m$, then the term $f(\mathbf{t})$ is in $\mathbb{F}_0$;
(iii) only terms obtained by the first two rules are in $\mathbb{F}_0$.

Then form the set $\mathbb{R}_0$ of all predicate symbols on $\mathbb{F}_0$, i.e.,

(iv) if $r \in R_\tau$ has arity n and $\mathbf{u} \in \mathbb{F}_0^n$, then the term $r(\mathbf{u})$ is in $\mathbb{R}_0$;
(v) only these terms are in $\mathbb{R}_0$.

The members of $\mathbb{R}_0$ are the *atomic formulae* of our system.

6. Form the term algebra $\mathbb{F}_1 = W_\lambda(\mathbb{R}_0)$, i.e., the free closure of $\mathbb{R}_0 \cup C_\lambda$ under the function symbols of λ. Thus,

(i) $\mathbb{R}_0 \cup C_\lambda \subseteq \mathbb{F}_1$;
(ii) if $g \in F_\lambda$ has arity m and $\mathbf{t} \in \mathbb{F}_1^m$, then the term $g(\mathbf{t})$ is in $\mathbb{F}_1$;
(iii) only terms obtained by the first two rules are in $\mathbb{F}_1$.

Thus the operation symbols of F_λ serve as logical connectives. The elements of $\mathbb{F}_1$ are *propositions* at the first level.

A similar, but distinct, role is played by the relation symbols of R_λ. Let $\mathbb{R}_1$ be the set of all expressions corresponding to relations on $\mathbb{F}_1$ in the type λ, i.e.,

(iv) if $s \in R_\lambda$ has arity n and $\mathbf{u} \in \mathbb{F}_1^n$, then the term $s(\mathbf{u})$ is in $\mathbb{R}_1$;
(v) only these terms are in $\mathbb{R}_1$.

The elements of $\mathbb{R}_1$ (and more generally, its operational closure $\mathbb{F}_2$) are propositions at the second level. They express relations between propositions in $\mathbb{F}_1$.

Alternatively, in place of $\mathbb{F}_1$ we could use the free algebra $\mathbf{F}_{\mathcal{V}}(\mathbb{R}_0)$ for any variety $\mathcal{V}$ containing $\mathbf{A}_\tau$. This refinement can of course be useful in practice, as for example when we implicitly use the commutativity and associativity of logical conjunction.

7. A *truth assignment* is a function $\varphi : \mathbb{R}_0 \to \mathbf{A}_\tau$. A truth assignment can be recursively extended to a homomorphism (still denoted by φ to conserve notation) $\varphi : \mathbb{F}_1 \to \mathbf{A}_\tau$. For a term $g(\mathbf{t})$ with $g \in F_\lambda$ and $\mathbf{t} \in \mathbb{F}_1^m$, where $m = \alpha_\lambda(g)$ and such that $\varphi(\mathbf{t}_i)$ is already defined for $1 \le i \le m$, define $\varphi(g(\mathbf{t})) = g^{\mathbf{A}_\tau}(\varphi(\mathbf{t}))$.

The truth function $\varphi : \mathbb{F}_0 \to \mathbf{A}_\tau$ likewise induces a truth function for relations, so that $\overline{\varphi} : \mathbb{R}_1 \to \mathbf{A}_\lambda$ by the rule $\overline{\varphi}(s(\mathbf{u})) = s^{\mathbf{A}_\tau}(\varphi(\mathbf{u}))$, whenever $s \in R_\lambda$ and $\mathbf{u} \in \mathbb{F}_1^n$ with $n = \alpha_\lambda(s)$. Note that $\overline{\varphi}(s(\mathbf{u}))$ is in $\mathbf{A}_\lambda$.

What we usually call the *propositional calculus* consist of replacing $\mathbb{R}_0$ by a set Z of propositional variables, and then evaluating functions in $\mathrm{Hom}(W_\lambda(Z), \mathbf{A}_\tau)$, i.e., determining when two terms from $W_\lambda(Z)$ are equal for all evaluations in $\mathbf{A}_\tau$, which is to say in $\mathbf{F}_{\mathcal{V}(\mathbf{A}_\tau)}(Z)$, the free algebra in the variety generated by $\mathbf{A}_\tau$. Thus we use standard truth tables for determining equality of terms in $\mathbf{F}_{\mathbf{BA}}(n)$, corresponding to the case when $\mathbf{A}_\tau$ is the two-element Boolean algebra. It is an interesting exercise to try the analogue of this process for other choices of $\mathbf{A}_\tau$.

Before proceeding, let pause for a brief discussion. Thusfar we have been dealing with the syntax of our logic. That is, we have considered the algebra of truth functions, but not their interpretation in specific τ-structures. Figure 2 illustrates the syntactic structure for the case when $\mathbf{A}_\tau$ has both functions and relations, but $\mathbf{A}_\lambda$ is an algebra with no relations. If $\mathbf{A}_\tau$ had no relations, as is often the case, then only the first level would be there. While further extensions are possible and potentially useful, the aspect that we want to emphasize is the diversity of algebras that could be, and are, used for $\mathbf{A}_\tau$, $\mathbf{A}_\lambda$, etc. The multiple layers of syntax are of secondary importance.

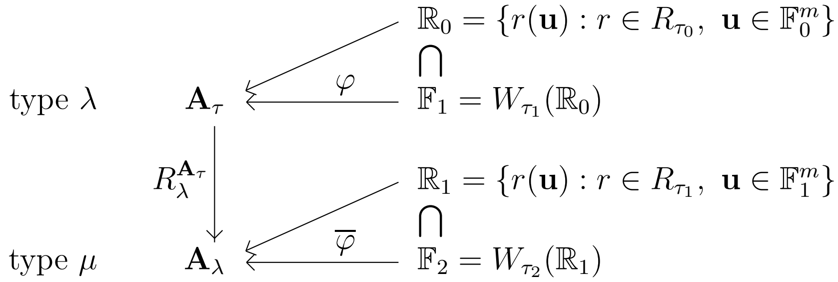

FIGURE 2

Presumably, the language is given. For example, the language of set theory might include a countable set $X = \{x_0, x_1, \dots\}$ of variables, $C = \{\emptyset\}$ so that the empty set is a constant, relations $R = \{\in, \approx\}$ for membership and equality, and no proper function symbols.

Our main choice is in the selection of $\mathbf{A}_\tau$. Depending on its type λ, $\mathbf{A}_\tau$ could be a set, an algebra, or a proper relational structure. Possibilities include any given ordered set, semilattice, lattice, lattice with operators, Boolean algebra, etc. These sort of structures give us the standard variations on logic. For example, we use the real interval $[0, 1]$ to get fuzzy logic. An interesting exercise is to construct the algebra on the set $\{T, U, F\}$, where U corresponds to the possibility "undecidable", using the standard logical operations $\wedge$, $\vee$ and $\neg$.

But $\mathbf{A}_\tau$ need not have an apparent logical meaning. We could choose it to be any fixed group $\mathbf{G}$, or indeed any other concrete algebra. An interesting choice is the *jan-ken-po* relational structure $\mathbf{J} = \{\{R, P, S\}, <\}$ with the relation $R < P < S < R$, or the corresponding algebra $\mathbf{J}^* = \{\{R, P, S\}, *\}$ with a commutative, idempotent multiplication given by the following table.

$*$	R	P	S
R	R	P	R
P	P	P	S
S	R	S	S

Or, losing idempotence, we could consider the quasi-group $\mathbf{J}^\circ$ with the following operation.

$\circ$	R	P	S
R	S	P	R
P	P	R	S
S	R	S	P

The word problem for free algebras in the variety generated by $\mathbf{A}$ is always of interest. In particular, it would be worthwhile to attempt this for the algebras $\mathbf{J}^*$ and $\mathbf{J}^\circ$.

In ordinary logic, with $\mathbf{A}_\tau$ being the Boolean algebra $\mathbf{2}$, we are able to avoid directly considering the relation $\leq$ by using the implication operation $\rightarrow$. In general, however, we must deal with relations on $\mathbf{A}_\tau$. This adds another level to our syntax structure.

Now we turn to semantics, i.e., applying the logic to describe a τ-structure $\mathbf{S}$.

8. An *interpretation* is a mapping $\sigma_0 : X \to \mathbf{S}$, where $\mathbf{S}$ is a structure of type τ. Each interpretation determines a truth assignment in a natural way. The map σ_0 can be extended to a homomorphism on the term algebra $W_\tau(X)$, so we obtain $\sigma : \mathbb{F}_0 \to \mathbf{S}$. The truth function $\widehat{\sigma} : \mathbb{R}_0 \to \mathbf{A}_\tau$ is determined by the rule that if $r \in R_\tau$ has arity n and $\mathbf{u} \in \mathbb{F}_0^n$, then $\widehat{\sigma}(r(\mathbf{u})) = r^{\mathbf{S}}(\sigma(\mathbf{u}))$. This can in turn be extended to $\mathbb{F}_1$ and $\mathbb{R}_1$ as indicated in subsection (7).

With interpretations, it is convenient to extend the language to include the elements of $\mathbf{S}$ as constants. This allows us to consider propositions such as $f(x, s)$, with $x \in X$ a variable and $s \in \mathbf{S}$ a fixed element. In the discussion of quantifiers below, we will implicitly assume that this has been done.

9. Next we add *quantifiers*. The purpose of quantifiers is to eliminate free variables, turning statements whose truth value depends on the substitution of variables into sentences to which we can assign a truth value. To do this, we extend our logical language by quantifier symbols Qx, and we need to determine how to extend truth functions, or at least those arising from interpretations. That is, given a τ-structure $\mathbf{S}$, an interpretation $\sigma_0 : X \to \mathbf{S}$, and a formula P involving quantifiers, how do we determine the extension $\widehat{\sigma}(Qx\ P)$? The details can be organized as follows.

Let $\mathbb{Q}_1$ be a set of quantifier symbols. Fix the set of variables X. The extension $\mathbb{E}_1$ of $\mathbb{F}_1 = W_\lambda(\mathbb{R}_0)$ is formed using these rules.

(i) $\mathbb{R}_0 \subseteq \mathbb{E}_1$.
(ii) If $g \in F_\lambda$ has arity m and $\mathbf{t} \in \mathbb{E}_1^m$, then the term $g(\mathbf{t})$ is in $\mathbb{E}_1$.
(iii) If $Q \in \mathbb{Q}_1$, $x \in X$ and $P \in \mathbb{E}_1$, then $Qx\ P \in \mathbb{E}_1$.
(iv) Only terms obtained by the first three rules are in $\mathbb{E}_1$.

Clearly, $\mathbb{R}_0 \subseteq \mathbb{F}_1 \subseteq \mathbb{E}_1$. Let us use the suggestive notation $\mathbb{E}_1 = W_{\tau_1 \cup \mathbb{Q}_1}(\mathbb{R}_0)$.

For $P \in \mathbb{E}_1$, define the *variables occurring in* P, the *scope* of a quantifier Q in P, and the *free occurrences* and *bound occurrences* of a variable in the usual way.

Now suppose we have a τ-structure $\mathbf{S}$ and an interpretation $\sigma_0 : X \to \mathbf{S}$. The interpretation $\widehat{\sigma}$ is already defined on $\mathbb{F}_1$, and we recursively extend it to the quantified formulas of $\mathbb{E}_1$, mimicing the standard procedure as in Enderton [7]. For any interpretation $\rho_0 : X \to \mathbf{S}$, any variable $x \in X$ and any element $s \in \mathbf{S}$, we define another interpretation $\rho_0(x|s) : X \to \mathbf{S}$ by

$$\rho_0(x|s)(y) = \begin{cases} \rho_0(y) & \text{if } y \neq x, \\ s & \text{if } y = x. \end{cases}$$

Each interpretation $\xi_0 : X \to \mathbf{A}_\tau$ can be extended to $\widehat{\xi} : \mathbb{F}_1 \to \mathbf{A}_\tau$ as before.

Corresponding to every quantifier symbol Q, there should be designated a function $f_Q : \mathbf{A}_\tau^S \to \mathbf{A}_\tau$, where $\mathbf{A}_\tau^S = \prod_{s \in S} \mathbf{A}_\tau$. The function f_Q determines the meaning of the quantifier Q. The rules for extension are as follows.

(ii) If $g \in F_\lambda$ and $\mathbf{t} \in \mathbb{E}_0^m$ and $\widehat{\sigma}(\mathbf{t}_i)$ has been defined for $1 \leq i \leq m$, then $\widehat{\sigma}(g(\mathbf{t})) = g^{\mathbf{S}}(\widehat{\sigma}(\mathbf{t}))$.
(iii) If $Q \in \mathbb{Q}_1$, $x \in X$, $P \in \mathbb{E}_0$, and $\widehat{\rho}(P)$ has been defined for an arbitrary interpretation ρ_0, then $\widehat{\sigma}(Qx\ P) = f_Q(\langle \widehat{\sigma(x|s)}(P)) : s \in \mathbf{S}\rangle)$.

Note that this just formalizes the idea that $\widehat{\sigma(x|s)}(P)$ is obtained by replacing every free occurrence of x in P by s, and the meaning of the quantifier is some predetermined function of the value of all those substitutions. It is not clear what restrictions we want to put on the function f_Q, if any, except that the interpretation of a given quantifier symbol Q should have the same meaning (in some sense) on different τ-structures $\mathbf{S}$.

Two examples will suffice for now. If the algebra $\mathbf{A}_\tau$ has a complete semilattice operation, then we can define

$$\widehat{\sigma}(\forall x\ P) = \bigwedge_{s \in S} \widehat{\sigma(x|s)}(P).$$

At the other extreme, if the type τ has constants, then for any $c \in C_\tau$ we can define a quantifier S_c such that

$$\widehat{\sigma}(S_c x\ P) = \widehat{\sigma(x|c^{\mathbf{S}})}(P).$$

Other kinds of quantifiers would include various versions of *almost everywhere* or *almost nowhere*. However, if the operations of $\mathbf{A}_\tau$ are different from those we are accustomed to in logic, then quantifiers may have quite different meanings.

10. We say that two formulas P, $R \in \mathbb{E}_1$ are *equivalent*, written $P \equiv R$, if $\widehat{\sigma}(P) = \widehat{\sigma}(R)$ for every interpretation $\widehat{\sigma}$ into a τ-structure. For example, in standard Boolean logic we have

$$[(\forall x \ r(x)) \to r(y)] \equiv 1$$

i.e., the left hand side is a tautology and hence a valid rule of inference. The business of *predicate logic* is to determine (if possible) the relation $\equiv$ on $\mathbb{E}_1$.

11. We can also define *satisfaction* in this general setting. Again let $\mathbf{S}$ be a τ-structure and $\sigma_0 : X \to \mathbf{S}$ an interpretation. For $P \in \mathbb{E}_0$ and $a \in \mathbf{A}_\tau$, we write

$$(\mathbf{S}, \sigma_0) \vDash (P, a) \quad \text{iff} \quad \widehat{\sigma}(P) = a.$$

This serves as a device to allow us to describe the properties of $\mathbf{S}$, or to discuss models of a collection of axioms of the form (P, a).

In the traditional setting, we write $(\mathbf{S}, \sigma_0) \vDash P$ in place of $(\mathbf{S}, \sigma_0) \vDash (P, 1)$, where 1 denotes the value *true*. When there are more options than just 0 and 1, we must be more specific.

The reader is strongly encouraged to work out examples of how various logics fit into the above scheme. The universal algebra viewpoint provides a good perspective for analyzing and comparing different logics.

It is clear that these ideas can be extended further. First of all, we have dealt primarily only with the first level of the general program indicated in Figure 1, but its recursive extension to more levels is straightforward. It is a fun exercise to concoct examples involving two or three levels of logical complexity. A more ambitious project would be to carry the analysis further and consider such notions as completeness, inference, and interpretability in a general setting.

The probability of any given person being abducted by aliens is fairly small. The prospect of increasing our understanding of our own mathematics and logic by considering alternative systems is considerably less remote. It is well known that learning a second language increases one's appreciation of the structure and subtlety of the first language. The same may be true of mathematics.

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