7. Varieties of Lattices

Variety is the spice of life.

A *lattice equation* is an expression $p \approx q$ where p and q are lattice terms. Our intuitive notion of what it means for a lattice $\mathcal{L}$ to satisfy $p \approx q$ (that $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$ whenever elements of $\mathcal{L}$ are substituted for the variables) is captured by the formal definition: $\mathcal{L}$ *satisfies* $p \approx q$ if $h(p) = h(q)$ for every homomorphism $h : W(X) \to \mathcal{L}$. We say that $\mathcal{L}$ satisfies a set Σ of equations if $\mathcal{L}$ satisfies every equation in Σ. As long as we are dealing entirely with lattices, there is no loss of generality in replacing p and q by the corresponding elements of $\mathrm{FL}(X)$, and in practice it is often more simple and natural to do so (as in Theorem 7.2 below).

A *variety* (or *equational class*) of lattices is the class of all lattices satisfying some set Σ of lattice equations. You are already familiar with several lattice varieties:

(1) the variety $\mathbf{T}$ of one-element lattices, satisfying $x \approx y$ (not very exciting);
(2) the variety $\mathbf{D}$ of distributive lattices, satisfying $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;
(3) the variety $\mathbf{M}$ of modular lattices, satisfying $(x \vee y) \wedge (x \vee z) \approx x \vee (z \wedge (x \vee y))$;
(4) the variety $\mathbf{L}$ of all lattices, satisfying $x \approx x$.

If $\mathbf{K}$ is any class of lattices, we say that a lattice $\mathcal{F}$ is $\mathbf{K}$-*freely generated* by its subset X if

(1) $\mathcal{F} \in \mathbf{K}$,
(2) X generates $\mathcal{F}$,
(3) for every lattice $\mathcal{L} \in \mathbf{K}$, every map $h_0 : X \to L$ can be extended to a homomorphism $h : \mathcal{F} \to \mathcal{L}$.

A lattice is $\mathbf{K}$-*free* if it is $\mathbf{K}$-freely generated by one of its subsets, and *relatively free* if it is $\mathbf{K}$-free for some (unspecified) class $\mathbf{K}$.

While these ideas floated around for some time before, it was Garrett Birkhoff [5] who proved the basic theorem about varieties in the 1930's.

Theorem 7.1. *If* $\mathbf{K}$ *is a nonempty class of lattices, then the following are equivalent.*

(1) $\mathbf{K}$ *is a variety.*
(2) $\mathbf{K}$ *is closed under the formation of homomorphic images, sublattices and direct products.*
(3) *Either* $\mathbf{K} = \mathbf{T}$ *(the variety of one-element lattices), or for every nonempty set* X *there is a lattice* $\mathcal{F}_{\mathbf{K}}(X)$ *which is* $\mathbf{K}$*-freely generated by* X*, and* $\mathbf{K}$ *is closed under homomorphic images.*

Proof. It is easy to see that varieties are closed under homomorphic images, sublattices and direct products, so (1) implies (2).

The crucial step in the equivalence, the construction of relatively free lattices $\mathcal{F}_{\mathbf{K}}(X)$, is a straightforward adaptation of the construction of FL(X). Let $\mathbf{K}$ be a class which is closed under the formation of sublattices and direct products, and let $\kappa = \bigcap\{\theta \in \mathbf{Con}\, W(X) : W(X)/\theta \in \mathbf{K}\}$. Following the proof of Theorem 6.1, we can show that $W(X)/\kappa$ is a subdirect product of lattices in $\mathbf{K}$, and that it is $\mathbf{K}$-freely generated by $\{x\kappa : x \in X\}$. Unless $\mathbf{K} = \mathbf{T}$, the classes $x\kappa$ $(x \in X)$ will be distinct. Thus (2) implies (3).

Finally, suppose that $\mathbf{K}$ is a class of lattices which is closed under homomorphic images and contains a $\mathbf{K}$-freely generated lattice $\mathcal{F}_{\mathbf{K}}(X)$ for every nonempty set X. For each nonempty X there is a homomorphism $f_X : W(X) \twoheadrightarrow \mathcal{F}_{\mathbf{K}}(X)$ which is the identity on X. Fix the countably infinite set $X_0 = \{x_1, x_2, x_3, \dots\}$, and let Σ be the collection of all equations $p \approx q$ such that $(p, q) \in \ker f_{X_0}$. Thus $p \approx q$ is in Σ if and only if $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$ in the countably generated lattice $\mathcal{F}_{\mathbf{K}}(X_0) \cong \mathcal{F}_{\mathbf{K}}(\omega)$.

Let $\mathbf{V}_\Sigma$ be the variety of all lattices satisfying Σ; we want to show that $\mathbf{K} = \mathbf{V}_\Sigma$. We formulate the critical argument as a sublemma.

Sublemma. *Let $\mathcal{F}_{\mathbf{K}}(Y)$ be a relatively free lattice. Let $p, q \in W(Y)$ and let $f_Y : W(Y) \twoheadrightarrow \mathcal{F}_{\mathbf{K}}(Y)$ with f_Y the identity on Y. Then $\mathbf{K}$ satisfies $p \approx q$ if and only if $f_Y(p) = f_Y(q)$.*

Proof. If $\mathbf{K}$ satisfies $p \approx q$, then $f_Y(p) = f_Y(q)$ because $\mathcal{F}_{\mathbf{K}}(Y) \in \mathbf{K}$. Conversely, if $f_Y(p) = f_Y(q)$, then by the mapping property (III) every lattice in $\mathbf{K}$ satisfies $p \approx q$.[1] □

Applying the Sublemma with $Y = X_0$, we conclude that $\mathbf{K}$ satisfies every equation of Σ, so $\mathbf{K} \subseteq \mathbf{V}_\Sigma$.

Conversely, let $\mathcal{L} \in \mathbf{V}_\Sigma$, and let X be a generating set for $\mathcal{L}$. The identity map on X extends to a surjective homomorphism $h : W(X) \twoheadrightarrow \mathcal{L}$, and we also have the map $f_X : W(X) \to \mathcal{F}_{\mathbf{K}}(X)$. For any pair $(p, q) \in \ker f_X$, the Sublemma says that $\mathbf{K}$ satisfies $p \approx q$. Again by the Sublemma, there is a corresponding equation in Σ (perhaps involving different variables). Since $\mathcal{L} \in \mathbf{V}_\Sigma$ this implies $h(p) = h(q)$. So $\ker f_X \le \ker h$, and hence by the Second Isomorphism Theorem there is a homomorphism $g : \mathcal{F}_{\mathbf{K}}(X) \to \mathcal{L}$ such that $h = gf_X$. Thus $\mathcal{L}$ is a homomorphic image of $\mathcal{F}_{\mathbf{K}}(X)$. Since $\mathbf{K}$ is closed under homomorphic images, this implies $\mathcal{L} \in \mathbf{K}$. Hence $\mathbf{V}_\Sigma \subseteq \mathbf{K}$, and equality follows. Therefore (3) implies (1). □

The three parts of Theorem 7.1 reflect three different ways of looking at varieties. The first is to start with a set Σ of equations, and to consider the variety $V(\Sigma)$ of all

[1] However, if Y is finite and $Y \subseteq Z$, then $\mathcal{F}_{\mathbf{K}}(Y)$ may satisfy equations not satisfied by $\mathcal{F}_{\mathbf{K}}(Z)$. For example, for any lattice variety, $\mathcal{F}_{\mathbf{K}}(2)$ is distributive. The Sublemma only applies to equations with at most $|Y|$ variables.

lattices satisfying those equations. The given equations will in general imply other equations, *viz.*, all the relations holding in the relatively free lattices $\mathcal{F}_{V(\Sigma)}(X)$. It is important to notice that while the proof of Birkhoff's theorem tells us abstractly how to construct relatively free lattices, it does not tell us how to solve the word problem for them. Consider the variety $\mathbf{M}$ of modular lattices. Richard Dedekind [6] showed in the 1890's that $\mathcal{F}_{\mathbf{M}}(3)$ has 28 elements; it is drawn in Figure 9.2. On the other hand, Ralph Freese [9] proved in 1980 that the word problem for $\mathcal{F}_{\mathbf{M}}(5)$ is unsolvable: *there is no algorithm for determining whether* $p = q$ *in* $\mathcal{F}_{\mathbf{M}}(5)$. Christian Herrmann [10] later showed that the word problem for $\mathcal{F}_{\mathbf{M}}(4)$ is also unsolvable. It follows, by the way, that the variety of modular lattices is not generated by its finite members:[2] *there is a lattice equation which holds in all finite modular lattices, but not in all modular lattices.*

Skipping to the third statement of Theorem 7.1, let $\mathbf{V}$ be a variety, and let κ be the kernel of the natural homomorphism $h : \mathrm{FL}(X) \to \mathcal{F}_{\mathbf{V}}(X)$ with $h(x) = x$ for all $x \in X$. Then, of course, $\mathcal{F}_{\mathbf{V}}(X) \cong \mathrm{FL}(X)/\kappa$. We want to ask which congruences on $\mathrm{FL}(X)$ arise in this way, i.e., *for which* $\theta \in \mathbf{Con}\ \mathrm{FL}(X)$ *is* $\mathrm{FL}(X)/\theta$ *relatively free?* To answer this, we need a couple of definitions.

An *endomorphism* of a lattice $\mathcal{L}$ is a homomorphism $f : \mathcal{L} \to \mathcal{L}$. The set of endomorphisms of $\mathcal{L}$ forms a semigroup $\mathbf{End}\ \mathcal{L}$ under composition. It is worth noting that an endomorphism of a lattice is determined by its action on a generating set, since $f(p(x_1, \ldots, x_n) = p(f(x_1), \ldots, f(x_n))$ for any lattice term p. In particular, an endomorphism f of $\mathrm{FL}(X)$ corresponds to a substitution $x_i \mapsto f(x_i)$ of elements for the generators.

A congruence relation θ is *fully invariant* if $(x, y) \in \theta$ implies $(f(x), f(y)) \in \theta$ for every endomorphism f of $\mathcal{L}$. The fully invariant congruences of $\mathcal{L}$ can be thought of as the congruence relations of the algebra $\mathcal{L}^* = (L, \wedge, \vee, \{f : f \in \mathrm{End}\ \mathcal{L}\})$. In particular, they form an algebraic lattice, in fact a complete sublattice of $\mathbf{Con}\,\mathcal{L}$.

The answer to our question, in these terms, is again due to Garrett Birkhoff [4].

Theorem 7.2. $\mathrm{FL}(X)/\theta$ *is relatively freely generated by* $\{x\theta : x \in X\}$ *if and only if* θ *is fully invariant.*

Proof. Let $\mathbf{V}$ be a lattice variety and let $h : \mathrm{FL}(X) \to \mathcal{F}_{\mathbf{V}}(X)$ with $h(x) = x$ for all $x \in X$. Then $h(p) = h(q)$ if and only if $\mathbf{V}$ satisfies $p \approx q$ (as in the Sublemma). Hence, for any endomorphism f and elements p, $q \in \mathrm{FL}(X)$, if $h(p) = h(q)$ then

$$\begin{aligned} hf(p) = h(f(p(x_1, \ldots, x_n))) &= h(p(f(x_1), \ldots, f(x_n))) \\ &= h(q(f(x_1), \ldots, f(x_n))) \\ &= h(f(q(x_1, \ldots, x_n))) = hf(q) \end{aligned}$$

[2]If a variety $\mathbf{V}$ of algebras (1) has only finitely many operation symbols, (2) is finitely based, and (3) is generated by its finite members, then the word problem for $\mathcal{F}_{\mathbf{V}}(X)$ is solvable. This result is due to A. I. Malcev for groups; see T. Evans [7].

so that $(f(p), f(q) \in \ker h$. Thus $\ker h$ is fully invariant.

Conversely, assume that θ is a fully invariant congruence on $\mathrm{FL}(X)$. If $\theta = 1_{\mathbf{Con}\ \mathrm{FL}(X)}$, then θ is fully invariant and $\mathrm{FL}(X)/\theta$ is relatively free for the trivial variety $\mathbf{T}$. So without loss of generality, θ is not the universal relation. Let $k : \mathrm{FL}(X) \to \mathrm{FL}(X)/\theta$ be the canonical homomorphism with $\ker k = \theta$. Let $\mathbf{V}$ be the variety determined by the set of equations $\Sigma = \{p \approx q : (p, q) \in \theta\}$. To show that $\mathrm{FL}(X)/\theta$ is $\mathbf{V}$-freely generated by $\{x\theta : x \in X\}$, we must verify that

(1) $\mathrm{FL}(X)/\theta \in \mathbf{V}$, and
(2) if $\mathcal{M} \in \mathbf{V}$ and $h_0 : X \to M$, then there is a homomorphism $h : \mathrm{FL}(X)/\theta \to \mathcal{M}$ such that $h(x\theta) = h_0(x)$, i.e., $hk(x) = h_0(x)$ for all $x \in X$.

For (1), we must show that the lattice $\mathrm{FL}(X)/\theta$ satisfies every equation of Σ, i.e., that if $p(x_1, \dots, x_n)\ \theta\ q(x_1, \dots, x_n)$ and $w_1, \dots, w_n$ are elements of $\mathrm{FL}(X)$, then $p(w_1, \dots, w_n)\ \theta\ q(w_1, \dots, w_n)$. Since there is an endomorphism f of $\mathrm{FL}(X)$ with $f(x_i) = w_i$ for all i, this follows from the fact that θ is fully invariant.

To prove (2), let $g : \mathrm{FL}(X) \to \mathcal{M}$ be the homomorphism such that $g(x) = h_0(x)$ for all $x \in X$. Since $\mathcal{M}$ is in $\mathbf{V}$, $g(p) = g(q)$ whenever $p \approx q$ is in Σ, and thus $\theta = \ker k \le \ker g$. By the Second Isomorphism Theorem, there is a homomorphism $h : \mathrm{FL}(X)/\theta \to \mathcal{M}$ such that $hk = g$, as desired. □

It follows that varieties of lattices are in one-to-one correspondence with fully invariant congruences on $\mathrm{FL}(\omega)$. The consequences of this fact can be summarized as follows.

Theorem 7.3. *The set of all lattice varieties ordered by containment forms a lattice Λ dually isomorphic to the lattice of all fully invariant congruences of* $\mathrm{FL}(\omega)$. *Thus Λ is dually algebraic, and a variety $\mathbf{V}$ is dually compact in Λ if and only if $\mathbf{V} = V(\Sigma)$ for some finite set of equations Σ.*

Going back to statement (2) of Theorem 7.1, the third way of looking at varieties is model theoretic: a variety is a class of lattices closed under the operators H (homomorphic images), S (sublattices) and P (direct products). Now elementary arguments show that, for any class $\mathbf{K}$,

$$\mathrm{PS}(\mathbf{K}) \subseteq \mathrm{SP}(\mathbf{K})$$
$$\mathrm{PH}(\mathbf{K}) \subseteq \mathrm{HP}(\mathbf{K})$$
$$\mathrm{SH}(\mathbf{K}) \subseteq \mathrm{HS}(\mathbf{K}).$$

Thus the smallest variety containing a class $\mathbf{K}$ of lattices is $\mathrm{HSP}(\mathbf{K})$, the class of all homomorphic images of sublattices of direct products of lattices in $\mathbf{K}$. We refer to $\mathrm{HSP}(\mathbf{K})$ as the variety *generated by* $\mathbf{K}$. We can think of HSP as a closure operator, but not an algebraic one: Λ is not upper continuous, so it cannot be algebraic (see Exercise 5). The many advantages of this point of view will soon become apparent.

Lemma 7.4. *Two lattice varieties are equal if and only if they contain the same subdirectly irreducible lattices.*

Proof. Recall from Theorem 5.6 that every lattice $\mathcal{L}$ is a subdirect product of subdirectly irreducible lattices $\mathcal{L}/\theta$ with θ completely meet irreducible in $\mathbf{Con}\,\mathcal{L}$. Suppose $\mathbf{V}$ and $\mathbf{K}$ are varieties, and that the subdirectly irreducible lattices of $\mathbf{V}$ are all in $\mathbf{K}$. Then for any X the relatively free lattice $\mathcal{F}_{\mathbf{V}}(X)$, being a subdirect product of subdirectly irreducible lattices $\mathcal{F}_{\mathbf{V}}(X)/\theta$ in $\mathbf{V}$, is a subdirect product of lattices in $\mathbf{K}$. Hence $\mathcal{F}_{\mathbf{V}}(X) \in \mathbf{K}$ and $\mathbf{V} \subseteq \mathbf{K}$. The lemma follows by symmetry. □

This leads us directly to a crucial question: *If* $\mathbf{K}$ *is a set of lattices, how can we find the subdirectly irreducible lattices in* $\mathrm{HSP}(\mathbf{K})$*?* The answer, due to Bjarni Jónsson, requires that we once again venture into the world of logic.

Let us recall that a *filter* (or *dual ideal*) of a lattice $\mathcal{L}$ with greatest element 1 is a subset F of L such that

(1) $1 \in F$,
(2) $x, y \in F$ implies $x \wedge y \in F$,
(3) $z \geq x \in F$ implies $z \in F$.

For any $x \in L$, the set $1/x$ is called a *principal filter*. As an example of a nonprincipal filter, in the lattice $\mathfrak{P}(X)$ of all subsets of an infinite set X we have the filter F of all complements of finite subsets of X. A maximal proper filter is called an *ultrafilter*.

We want to describe an important type of congruence relation on direct products. Let $\mathcal{L}_i$ $(i \in I)$ be lattices, and let F be a filter on the lattice of subsets $\mathfrak{P}(I)$. We define an equivalence relation $\equiv_F$ on the direct product $\prod_{i \in I} \mathcal{L}_i$ by

$$x \equiv_F y \text{ if } \{i \in I : x_i = y_i\} \in F.$$

A routine check shows that $\equiv_F$ is a congruence relation.

Lemma 7.5. *(1) Let $\mathcal{L}$ be a lattice, F a filter on $\mathcal{L}$, and $a \notin F$. Then there exists a filter G on $\mathcal{L}$ maximal with respect to the properties $F \subseteq G$ and $a \notin G$.*

(2) A proper filter U on $\mathfrak{P}(I)$ is an ultrafilter if and only if for every $A \subseteq I$, either $A \in U$ or $I - A \in U$.

(3) If U is an ultrafilter on $\mathfrak{P}(I)$, then its complement $\mathfrak{P}(I) - U$ is a maximal proper ideal.

(4) If U is an ultrafilter and $A_1 \cup \cdots \cup A_n \in U$, then $A_i \in U$ for some i.

(5) An ultrafilter U is nonprincipal if and only if it contains the filter of all complements of finite subsets of I.

Proof. Part (1) is a straightforward Zorn's Lemma argument. Moreover, it is clear that a proper filter U is maximal if and only if for every $A \notin U$ there exists $B \in U$ such that $A \cap B = \emptyset$, i.e., $B \subseteq I - A$. Thus U is an ultrafilter if and only if $A \notin U$ implies $I - A \in U$, which is (2). DeMorgan's Laws then yield (3), which in turn implies (4). It follows from (4) that if an ultrafilter U on I contains a finite set, then it contains a singleton $\{i_0\}$, and hence is principal with $U = 1/\{i_0\} = \{A \subseteq I : i_0 \in A\}$. Conversely, if U is a principal ultrafilter $1/S$, then S must be a singleton. Thus an ultrafilter is nonprincipal if and only if it contains no finite set, which by (2) means that it contains the complement of every finite set. $\square$

Corollary. *If I is an infinite set, then there is a nonprincipal ultrafilter on $\mathfrak{P}(I)$.*

Proof. Apply Lemma 7.5(1) with $\mathcal{L} = \mathfrak{P}(I)$, F the filter of all complements of finite subsets of I, and $a = \emptyset$. $\square$

If F is a filter on $\mathfrak{P}(I)$, the quotient lattice $\prod_{i\in I} \mathcal{L}_i/\equiv_{\mathrm{F}}$ is called a *reduced product*. If U is an ultrafilter, then $\prod_{i\in I} \mathcal{L}_i/\equiv_{\mathrm{U}}$ is an *ultraproduct*. The interesting case is when U is a nonprincipal ultrafilter. Good references on reduced products and ultraproducts are [3] and [8].

Our next immediate goal is to investigate what properties are preserved by the ultraproduct construction. In order to be precise, we begin with a slough of definitions, reserving comment for later.

The elements of a *first order language* for lattices are

(1) a countable alphabet $X = \{x_1, x_2, x_3, \dots\}$,
(2) equations $p \approx q$ with $p, q \in W(X)$,
(3) logical connectives AND, OR, and $\neg$,
(4) quantifiers $\forall x_i$ and $\exists x_i$ for $i = 1, 2, 3 \dots$.

These symbols can be combined appropriately to form *well formed formulas* (wffs) by the following rules.

(1) Every equation $p \approx q$ is a wff.
(2) If α and β are wffs, then so are $(\neg\alpha)$, $(\alpha \text{ AND } \beta)$ and $(\alpha \text{ OR } \beta)$.
(3) If γ is a wff and $i \in \{1, 2, 3, \dots\}$, then $\forall x_i \gamma$ and $\exists x_i \gamma$ are wffs.
(4) Only expressions generated by the first three rules are wffs.

Now let $\mathcal{L}$ be a lattice, let $h : W(X) \to \mathcal{L}$ be a homomorphism, and let φ be a well formed formula. We say that the pair $(\mathcal{L}, h)$ *models* φ, written symbolically as $(\mathcal{L}, h) \models \varphi$, according to the following recursive definition.

(1) $(\mathcal{L}, h) \models p \approx q$ if $h(p) = h(q)$, i.e., if $p(h(x_1), \dots, h(x_n)) = q(h(x_1), \dots, h(x_n))$.
(2) $(\mathcal{L}, h) \models (\neg\alpha)$ if $(\mathcal{L}, h)$ does not model α (written $(\mathcal{L}, h) \not\models \alpha$).
(3) $(\mathcal{L}, h) \models (\alpha \text{ AND } \beta)$ if $(\mathcal{L}, h) \models \alpha$ and $(\mathcal{L}, h) \models \beta$.
(4) $(\mathcal{L}, h) \models (\alpha \text{ OR } \beta)$ if $(\mathcal{L}, h) \models \alpha$ or $(\mathcal{L}, h) \models \beta$ (or both).
(5) $(\mathcal{L}, h) \models \forall x_i \gamma$ if $(\mathcal{L}, g) \models \gamma$ for every g such that $g|_{X-\{x_i\}} = h|_{X-\{x_i\}}$.
(6) $(\mathcal{L}, h) \models \exists x_i \gamma$ if $(\mathcal{L}, g) \models \gamma$ for some g such that $g|_{X-\{x_i\}} = h|_{X-\{x_i\}}$.

(For $Y \subseteq X$, $g|_Y$ denotes the restriction of g to Y.)

We say that $\mathcal{L}$ *satisfies* φ (or $\mathcal{L}$ *models* φ) if $(\mathcal{L}, h)$ models φ for every homomorphism $h : W(X) \to \mathcal{L}$.

We are particularly interested in well formed formulas φ for which all the variables appearing in φ are quantified (by $\forall$ or $\exists$). The set F_φ of variables that *occur freely* *in* φ is defined recursively as follows.

(1) For an equation, $F_{p \approx q}$ is the set of all variables x_i which actually appear in p or q.
(2) $F_{\neg\alpha} = F_\alpha$.
(3) $F_{\alpha \text{ AND } \beta} = F_\alpha \cup F_\beta$.
(4) $F_{\alpha \text{ OR } \beta} = F_\alpha \cup F_\beta$.
(5) $F_{\forall x_i \alpha} = F_\alpha - \{x_i\}$.
(6) $F_{\exists x_i \alpha} = F_\alpha - \{x_i\}$.

A *first order sentence* is a well formed formula φ such that F_φ is empty, i.e., no variable occurs freely in φ. It is not hard to show inductively that, for a given lattice $\mathcal{L}$ and any well formed formula φ, whether or not $(\mathcal{L}, h) \models \varphi$ depends only on the values of $h|_{F_\varphi}$, i.e., if $g|_{F_\varphi} = h|_{F_\varphi}$, then $(\mathcal{L}, g) \models \varphi$ iff $(\mathcal{L}, h) \models \varphi$. So if φ is a sentence, then either $\mathcal{L}$ satisfies φ or $\mathcal{L}$ satisfies $\neg\varphi$.

Now some comments are in order. First of all, we did not include the predicate $p \leq q$ because we can capture it with the equation $p \vee q \approx q$. Likewise, the logical connective $\Longrightarrow$ is omitted because $(\alpha \Longrightarrow \beta)$ is equivalent to $(\neg\alpha)$ OR β. On the other hand, our language is redundant because OR can be eliminated by the use of DeMorgan's law, and $\exists x_i \varphi$ is equivalent to $\neg\forall x_i(\neg\varphi)$.

Secondly, for any well formed formula φ, a lattice $\mathcal{L}$ satisfies φ if and only if it satisfies the sentence $\forall x_{i_1} \ldots \forall x_{i_k} \varphi$ where the quantification runs over the variables in F_φ. Thus we can consistently speak of a lattice satisfying an equation or Whitman's condition, for example, when what we really have in mind is the corresponding universally quantified sentence.

Fortunately, our intuition about what sort of properties can be expressed as first order sentences, and what it means for a lattice to satisfy a sentence φ, tends to be pretty good, particularly after we have seen a lot of examples. With this in mind, let us list some first order properties.

(1) $\mathcal{L}$ satisfies $p \approx q$.
(2) $\mathcal{L}$ satisfies the semidistributive laws $(SD_\vee)$ and $(SD_\wedge)$.
(3) $\mathcal{L}$ satisfies Whitman's condition (W).
(4) $\mathcal{L}$ has width 7.
(5) $\mathcal{L}$ has at most 7 elements.
(6) $\mathcal{L}$ has exactly 7 elements.
(7) $\mathcal{L}$ is isomorphic to $\mathcal{M}_5$.

And, of course, we can do negations and finite conjunctions and disjunctions of these. The sort of things which *cannot* be expressed by first order sentences includes the following.

(1) $\mathcal{L}$ is finite.
(2) $\mathcal{L}$ satisfies the ACC.
(3) $\mathcal{L}$ has finite width.
(4) $\mathcal{L}$ is subdirectly irreducible.

Now we are in a position to state for lattices the fundamental theorem about ultraproducts, due to J. Łos in 1955 [13].

Theorem 7.6. *Let φ be a first order lattice sentence, $\mathcal{L}_i$ $(i \in I)$ lattices, and U an ultrafilter on $\mathfrak{P}(I)$. Then the ultraproduct $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{U}}$ satisfies φ if and only if $\{i \in I : \mathcal{L}_i$ satisfies $\varphi\}$ is in U.*

Corollary. *If each $\mathcal{L}_i$ satisfies φ, then so does the ultraproduct $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{U}}$.*

Proof. Suppose we have a collection of lattices $\mathcal{L}_i$ $(i \in I)$ and an ultrafilter U on $\mathfrak{P}(I)$. The elements of the ultraproduct $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{U}}$ are equivalence classes of elements of the direct product. Let $\mu : \prod \mathcal{L}_i \to \prod \mathcal{L}_i / \equiv_{\mathrm{U}}$ be the canonical homomorphism, and let $\pi_j : \prod \mathcal{L}_i \to \mathcal{L}_j$ denote the projection map. We will prove the following claim, which includes Theorem 7.6.

Claim. *Let $h : W(X) \to \prod_{i \in I} \mathcal{L}_i$ be a homomorphism, and let φ be a well formed formula. Then $(\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \varphi$ if and only if $\{i \in I : (\mathcal{L}_i, \pi_i h) \models \varphi\} \in U$.*

We proceed by induction on the complexity of φ. In view of the observations above (e.g., DeMorgan's Laws), it suffices to treat equations, AND, $\neg$ and $\forall$. The first three are quite straightforward.

Note that for $a, b \in \prod \mathcal{L}_i$ we have $\mu(a) = \mu(b)$ if and only if $\{i : \pi_i(a) = \pi_i(b)\} \in U$. Thus, for an equation $p \approx q$, we have

$$\begin{aligned}(\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models p \approx q \quad &\text{iff} \quad \mu h(p) = \mu h(q) \\ &\text{iff} \quad \{i : \pi_i h(p) = \pi_i h(q)\} \in U \\ &\text{iff} \quad \{i : (\mathcal{L}_i, \pi_i h) \models p \approx q\} \in U.\end{aligned}$$

.

For a conjunction α AND β, using $A \cap B \in U$ iff $A \in U$ and $B \in U$, we have

$$\begin{aligned}(\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \alpha \text{ AND } \beta \; &\text{iff } (\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \alpha \text{ and } (\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \beta \\ &\text{iff } \{i : (\mathcal{L}_i, \pi_i h) \models \alpha\} \in U \text{ and } \{i : (\mathcal{L}_i, \pi_i h) \models \beta\} \in U \\ &\text{iff } \{i : (\mathcal{L}_i, \pi_i h) \models \alpha \text{ AND } \beta\} \in U.\end{aligned}$$

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For a negation $\neg\alpha$, using the fact that $A \in U$ iff $I - A \notin U$, we have

$$\begin{aligned}(\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \neg\alpha \quad &\text{iff} \quad (\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \not\models \alpha \\ &\text{iff} \quad \{i : (\mathcal{L}_i, \pi_i h) \models \alpha\} \notin U \\ &\text{iff} \quad \{j : (\mathcal{L}_j, \pi_j h) \not\models \alpha\} \in U \\ &\text{iff} \quad \{j : (\mathcal{L}_j, \pi_j h) \models \neg\alpha\} \in U.\end{aligned}$$

.

Finally, we consider the case when φ has the form $\forall x\gamma$. First, assume $A = \{i : (\mathcal{L}_i, \pi_i h) \models \forall x\gamma\} \in U$, and let $g : W(X) \to \prod \mathcal{L}_i$ be a homomorphism such that $\mu g|_{X-\{x\}} = \mu h|_{X-\{x\}}$. This means that for each $y \in X - \{x\}$, the set $B_y = \{j : \pi_j g(y) = \pi_j h(y)\} \in U$. Since F_γ is a finite set and U is closed under intersection, it follows that $B = \bigcap_{y \in F_\gamma - \{x\}} B_y = \{j : \pi_j g(y) = \pi_j h(y) \text{ for all } y \in F_\gamma - \{x\}\} \in U$. Therefore $A \cap B = \{i : (\mathcal{L}_i, \pi_i h) \models \forall x\gamma \text{ and } \pi_i g|_{F_\gamma - \{x\}} = \pi_i h|_{F_\gamma - \{x\}}\} \in U$. Hence $\{i : (\mathcal{L}_i, \pi_i g) \models \gamma\} \in U$, and so by induction $(\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu g) \models \gamma$. Thus $(\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \forall x\gamma$, as desired.

Conversely, suppose $A = \{i : (\mathcal{L}_i, \pi_i h) \models \forall x\gamma\} \notin U$. Then the complement $I - A = \{j : (\mathcal{L}_j, \pi_j h) \not\models \forall x\gamma\} \in U$. For each $j \in I - A$, there is a homomorphism $g_j : W(X) \to \mathcal{L}_j$ such that $g_j|_{X-\{x\}} = \pi_j h|_{X-\{x\}}$ and $(L_j, g_j) \not\models \gamma$. Let $g : W(X) \to \prod \mathcal{L}_i$ be a homomorphism such that $\pi_j g = g_j$ for all $j \in I - A$. Then $\mu g|_{X-\{x\}} = \mu h|_{X-\{x\}}$ but $(\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu g) \not\models \gamma$. Thus $(\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \not\models \forall x\gamma$.

This completes the proof of Lemma 7.6. □

To our operators H, S and P let us add a fourth: $\mathrm{P_u}(\mathbf{K})$ is the class of all ultraproducts of lattices from $\mathbf{K}$. Finally we get to answer the question: *Where do subdirectly irreducibles come from?*

Theorem 7.7. Jónsson's Lemma. *Let* $\mathbf{K}$ *be a class of lattices. If* $\mathcal{L}$ *is subdirectly irreducible and* $\mathcal{L} \in \mathrm{HSP}(\mathbf{K})$*, then* $\mathcal{L} \in \mathrm{HSP_u}(\mathbf{K})$.

Proof. Now $\mathcal{L} \in \mathrm{HSP}(\mathbf{K})$ means that there are lattices $\mathcal{K}_i \in \mathbf{K}$ $(i \in I)$, a sublattice $\mathcal{S}$ of $\prod_{i \in I} \mathcal{K}_i$, and a surjective homomorphism $h : \mathcal{S} \twoheadrightarrow \mathcal{L}$. If we also assume that $\mathcal{L}$ is finitely subdirectly irreducible (this suffices), then $\ker h$ is meet irreducible in $\mathbf{Con}\, \mathcal{S}$. Since $\mathbf{Con}\, \mathcal{S}$ is distributive, this makes $\ker h$ meet prime.

For any $J \subseteq I$, let π_J be the kernel of the projection of $\mathcal{S}$ onto $\prod_{j \in J} \mathcal{K}_j$. Thus for a, $b \in S$ we have $a \; \pi_J \; b$ iff $a_j = b_j$ for all $j \in J$. Note that $H \supseteq J$ implies $\pi_H \leq \pi_J$, and that $\pi_{J \cup K} = \pi_J \wedge \pi_K$.

Let $\mathfrak{H} = \{J \subseteq I : \pi_J \leq \ker h\}$. By the preceding observations,

(1) $I \in \mathfrak{H}$ and $\emptyset \notin \mathfrak{H}$,
(2) $\mathfrak{H}$ is an order filter in $\mathfrak{P}(I)$,
(3) $J \cup K \in \mathfrak{H}$ implies $J \in \mathfrak{H}$ or $K \in \mathfrak{H}$.

However, $\mathfrak{H}$ need not be a (lattice) filter. Let us therefore consider

$$\mathcal{Q} = \{F \subseteq \mathfrak{P}(I) : F \text{ is a filter on } \mathfrak{P}(I) \text{ and } F \subseteq \mathfrak{H}\}.$$

By Zorn's Lemma, $\mathcal{Q}$ contains a maximal member with respect to set inclusion, say U. Let us show that U is an ultrafilter.

If not, then by Lemma 7.5(2) there exists $A \subseteq I$ such that A and $I - A$ are both not in U. By the maximality of U, this means that there exists a subset $X \in U$ such that $A \cap X \notin \mathfrak{H}$. Similarly, there is a $Y \in U$ such that $(I - A) \cap Y \notin \mathfrak{H}$. Let

$Z = X \cap Y$. Then $Z \in U$, and hence $Z \in \mathfrak{H}$. However, $A \cap Z \subseteq A \cap X$, whence $A \cap Z \notin \mathfrak{H}$ by (2) above. Likewise $(I - A) \cap Z \notin \mathfrak{H}$. But

$$(A \cap Z) \cup ((I - A) \cap Z) = Z \in \mathfrak{H},$$

contradicting (3). Thus U is an ultrafilter.

Now $\equiv_U \in \mathbf{Con} \prod \mathcal{K}_i$, and its restriction is a congruence on $\mathcal{S}$. Moreover, $\mathcal{S}/\equiv_{\mathrm{U}}$ is (isomorphic to) a sublattice of $\prod \mathcal{K}_i/\equiv_{\mathrm{U}}$. If a, b are any pair of elements of $\mathcal{S}$ such that $a \equiv_U b$, then $J = \{i : a_i = b_i\} \in U$. This implies $J \in \mathfrak{H}$ and so $\pi_J \leq \ker h$, whence $h(a) = h(b)$. Thus the restriction of $\equiv_U$ to $\mathcal{S}$ is below $\ker h$, wherefore $\mathcal{L} = h(\mathcal{S})$ is a homomorphic image of $\mathcal{S}/\equiv_{\mathrm{U}}$. We conclude that $\mathcal{L} \in \mathrm{HSP_u}(\mathbf{K})$. □

The proof of Jónsson's Lemma [11] uses the distributivity of **Con** $\mathcal{L}$ in a crucial way, and its conclusion is not generally true for varieties of algebras which do not have distributive congruence lattices. This means that varieties of lattices are more well-behaved than varieties of other algebras, such as groups and rings. The applications below will indicate some aspects of this.

Lemma 7.8. *Let U be an ultrafilter on $\mathfrak{P}(I)$ and $J \in U$. Then $V = \{B \subseteq J : B \in U\}$ is an ultrafilter on $\mathfrak{P}(J)$, and $\prod_{j \in J} \mathcal{L}_j/\equiv_{\mathrm{V}}$ is isomorphic to $\prod_{i \in I} \mathcal{L}_i/\equiv_{\mathrm{U}}$.*

Proof. V is clearly a proper filter. Moreover, if $A \subseteq J$ and $A \notin V$, then $I - A \in U$ and hence $J - A = J \cap (I - A) \in U$. It follows by Lemma 7.5(2) that V is an ultrafilter.

The projection $\rho_J : \prod_{i \in I} \mathcal{L}_i \twoheadrightarrow \prod_{j \in J} \mathcal{L}_j$ is a surjective homomorphism. As $A \cap J \in U$ if and only if $A \in U$, it induces a (well defined) isomorphism of $\prod_{i \in I} \mathcal{L}_i/\equiv_{\mathrm{U}}$ onto $\prod_{j \in J} \mathcal{L}_j/\equiv_{\mathrm{V}}$. □

Theorem 7.9. *Let $\mathbf{K} = \{\mathcal{K}_1, \ldots, \mathcal{K}_n\}$ be a finite collection of finite lattices. If $\mathcal{L}$ is a subdirectly irreducible lattice in the variety* $\mathrm{HSP}(\mathbf{K})$, *then $\mathcal{L} \in \mathrm{HS}(\mathcal{K}_j)$ for some j.*

Proof. By Jónsson's Lemma, $\mathcal{L}$ is a homomorphic image of a sublattice of an ultraproduct $\prod_{i \in I} \mathcal{L}_i/\equiv_{\mathrm{U}}$ with each $\mathcal{L}_i$ isomorphic to one of $\mathcal{K}_1, \ldots, \mathcal{K}_n$. Let $A_j = \{i \in I : \mathcal{L}_i \cong \mathcal{K}_j\}$. As $A_1 \cup \cdots \cup A_n = I \in U$, by Lemma 7.5(4) there is a j such that $A_j \in U$. But then Lemma 7.8 says that there is an ultrafilter V on $\mathfrak{P}(A_j)$ such that the original ultraproduct is isomorphic to $\prod_{k \in A_j} \mathcal{L}_k/\equiv_{\mathrm{V}}$, wherein each $\mathcal{L}_k \cong K_j$. However, for any finite lattice $\mathcal{K}$ there is a first order sentence $\varphi_{\mathcal{K}}$ such that a lattice $\mathcal{M}$ satisfies $\varphi_{\mathcal{K}}$ if and only if $\mathcal{M} \cong \mathcal{K}$. Therefore, by Łos' Theorem, $\prod_{k \in A_j} \mathcal{L}_k/\equiv_{\mathrm{V}}$ is isomorphic to $\mathcal{K}_j$. Hence $\mathcal{L} \in HS(\mathcal{K}_j)$, as claimed. □

Corollary. *If $\mathbf{V} = \mathrm{HSP}(\mathbf{K})$ where $\mathbf{K}$ is a finite collection of finite lattices, then $\mathbf{V}$ contains only finitely many subvarieties.*

Note that $\mathrm{HSP}(\{\mathcal{K}_1, \ldots, \mathcal{K}_n\}) = \mathrm{HSP}(\mathcal{K}_1 \times \cdots \times \mathcal{K}_n)$, so w.l.o.g. we can talk about the variety generated by a single finite lattice. The author has recently shown that the converse of the Corollary is false [17]: *There is an infinite, subdirectly irreducible lattice $\mathcal{L}$ such that* HSP($\mathcal{L}$) *has only finitely many subvarieties, each of which is generated by a finite lattice.*

Let us call a variety $\mathbf{V}$ *finitely based* if $\mathbf{V} = V(\Sigma)$ for some finite set of equations Σ. These are just the varieties which are dually compact in the lattice Λ of lattice varieties. Ralph McKenzie [14] proved the following nice result.

Theorem 7.10. *The variety generated by a finite lattice is finitely based.*

Kirby Baker [1] generalized this result by showing that if $\mathcal{A}$ is any finite algebra in a variety $\mathbf{V}$ such that (i) $\mathbf{V}$ has only finitely many operation symbols, and (ii) the congruence lattices of algebras in $\mathbf{V}$ are distributive, then HSP($\mathcal{A}$) is finitely based. It is also true that the variety generated by a finite group is finitely based (S. Oates and M. B. Powell [18]), and likewise the variety generated by a finite ring (R. Kruse [12]). See R. McKenzie [15] for a common generalization of these finite basis theorems. There are many natural examples of finite algebras which do not generate a finitely based variety; see, e.g., G. McNulty [16].

We will return to the varieties generated by some particular finite lattices in the next chapter.

If $\mathbf{V}$ is a lattice variety, let $\mathbf{V}_{si}$ be the class of subdirectly irreducible lattices in $\mathbf{V}$. The next result is proved by a straightforward modification of the first part of the proof of Theorem 7.9.

Theorem 7.11. *If* $\mathbf{V}$ *and* $\mathbf{W}$ *are lattice varieties, then* $(\mathbf{V} \vee \mathbf{W})_{si} = \mathbf{V}_{si} \cup \mathbf{W}_{si}$.

Corollary. Λ *is distributive.*

Theorem 7.11 does not extend to infinite joins (finite lattices generate the variety of all lattices - see Exercise 5). We already knew the Corollary by Theorem 7.3, because Λ is dually isomorphic to a sublattice of **Con** FL(ω), which is distributive, but this provides an interesting way of looking at it.

In closing let us consider the lattice $\mathcal{I}(\mathcal{L})$ of ideals of $\mathcal{L}$. An elementary argument shows that the map $x \to x/0$ embeds $\mathcal{L}$ into $\mathcal{I}(\mathcal{L})$. A classic theorem of Garrett Birkhoff [4] says that $\mathcal{I}(\mathcal{L})$ satisfies every identity satisfied by $\mathcal{L}$, i.e., $\mathcal{I}(\mathcal{L}) \in$ HSP($\mathcal{L}$). The following result of Kirby Baker and Alfred Hales [2] goes one better.

Theorem 7.12. *For any lattice* $\mathcal{L}$, *we have* $\mathcal{I}(\mathcal{L}) \in \mathrm{HSP_u}(\mathcal{L})$.

This is an ideal place to stop.

Exercises for Chapter 7

1. Show that fully invariant congruences form a complete sublattice of **Con** $\mathcal{L}$.

2. Let $\mathcal{L}$ be a lattice and $\mathbf{V}$ a lattice variety. Show that there is a unique minimum congruence $\rho_{\mathbf{V}}$ on $\mathcal{L}$ such that $\mathcal{L}/\rho_{\mathbf{V}} \in \mathbf{V}$.

3. (a) Prove that if $\mathcal{L}$ is a subdirectly irreducible lattice, then HSP($\mathcal{L}$) is (finitely) join irreducible in the lattice Λ of lattice varieties.

(b) Prove that if a variety $\mathbf{V}$ is completely join irreducible in Λ, then $\mathbf{V} = \mathrm{HSP}(\mathcal{K})$ for some finitely generated, subdirectly irreducible lattice $\mathcal{K}$.

4. Show that if F is a filter on $\mathfrak{P}(I)$, then $\equiv_F$ is a congruence relation on $\prod_{i \in I} \mathcal{L}_i$.

5. Prove that every lattice equation which does not hold in all lattices fails in some finite lattice. (Let $p \neq q$ in $\mathrm{FL}(X)$. Then there exist a finite join subsemilattice $\mathcal{S}$ of $\mathrm{FL}(X)$ containing p, q and $0 = \bigwedge X$, and a lattice homomorphism $h : \mathrm{FL}(X) \to \mathcal{S}$, such that $h(p) = p$ and $h(q) = q$.)

The standard solution to Exercise 5 involves lattices which turn out to be lower bounded (see Exercise 11 of Chapter 6). Hence they satisfy $\mathrm{SD}_\vee$, and any finite collection of them generates a variety not containing $\mathcal{M}_3$, while all together they generate the variety of all lattices. On the other hand, the variety generated by $\mathcal{M}_3$ contains only the variety $\mathbf{D}$ of distributive lattices (generated by $\mathbf{2}$) and the trivial variety $\mathbf{T}$. It follows that the lattice Λ of lattice varieties is not join continuous.

6. Give a first order sentence characterizing each of the following properties of a lattice $\mathcal{L}$ (i.e., $\mathcal{L}$ has the property iff $\mathcal{L} \models \varphi$).

(a) $\mathcal{L}$ has a least element.
(b) $\mathcal{L}$ is atomic.
(c) $\mathcal{L}$ is strongly atomic.
(d) $\mathcal{L}$ is weakly atomic.
(e) $\mathcal{L}$ has no covering relations.

7. A lattice $\mathcal{L}$ has *breadth* n if L contains n elements whose join is irredundant, but every join of $n+1$ elements of L is redundant.

(a) Give a first order sentence characterizing lattices of breadth n (for a fixed finite integer $n \geq 1$).
(b) Show that the class of lattices of breadth $\leq n$ is not a variety.
(c) Show that a lattice $\mathcal{L}$ and its dual $\mathcal{L}^d$ have the same breadth.

8. Give a first order sentence φ such that a lattice $\mathcal{L}$ satisfies φ if and only if $\mathcal{L}$ is isomorphic to the four element lattice $\mathbf{2} \times \mathbf{2}$.

9. Prove Theorem 7.11.

10. Prove that $\mathcal{I}(\mathcal{L})$ is distributive if and only if $\mathcal{L}$ is distributive. Similarly, show that $\mathcal{I}(\mathcal{L})$ is modular if and only if $\mathcal{L}$ is modular.

References

1. K. Baker, *Finite equational bases for finite algebras in a congruence-distributive equational class*, Advances in Math. **24** (1977), 207–243.
2. K. Baker and A. W. Hales, *From a lattice to its ideal lattice*, Algebra Universalis **4** (1974), 250–258.
3. J. L. Bell and A. B. Slomson, *Models and Ultraproducts: an Introduction*, North-Holland, Amsterdam, 1971.
4. G. Birkhoff, *On the lattice theory of ideals*, Bull. Amer. Math. Soc. **40** (1934), 613–619.
5. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Phil. Soc. **31** (1935), 433–454.

6. R. Dedekind, *Über die drei Moduln erzeugte Dualgruppe*, Math. Annalen **53** (1900), 371–403.
7. T. Evans, *Some connections between residual finiteness, finite embeddability and the word problem*, J. London. Math. Soc. (2) **2** (1969), 399–403.
8. T. Frayne, A. Morel and D. Scott, *Reduced direct products*, Fund. Math. **51** (1962), 195–228.
9. R. Freese, *Free modular lattices*, Trans. Amer. Math. Soc. **261** (1980), 81–91.
10. C. Herrmann, *Uber die von vier Moduln erzeugte Dualgruppe*, Abh. Braunschweig. Wiss. Ges. **33** (1982), 157–159.
11. B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. **21** (1967), 110–121.
12. R. Kruse, *Identities satisfied by a finite ring*, J. Algebra **26** (1973), 298–318.
13. J. Łos, *Quelques remarques, théorèmes et problèmes sur le classes définissables d'algèbras*, Mathematical interpretations of formal systems, North-Holland, Amsterdam, 1955.
14. R. McKenzie, *Equational bases for lattice theories*, Math. Scand. **27** (1970), 24–38.
15. R. McKenzie, *Finite equational bases for congruence modular varieties*, Algebra Universalis **24** (1987), 224-250.
16. G. McNulty, *How to construct finite algebras which are not finitely based*, Universal Algebra and Lattice Theory (Charleston, S.C., 1984), Lecture Notes in Math., vol. 1149, Springer, Berlin-New York, 1985, pp. 167–174.
17. J. B. Nation, *A counterexample to the finite height conjecture*, Order **13** (1996), 1–9.
18. S. Oates and M. B. Powell, *Identical relations in finite groups*, J. Algebra **1** (1964), 11–39.

8. Distributive Lattices

Every dog must have his day.

In this chapter and the next we will look at the two most important lattice varieties: distributive and modular lattices. Let us set the context for our study of distributive lattices by considering varieties generated by a single finite lattice. A variety $\mathbf{V}$ is said to be *locally finite* if every finitely generated lattice in $\mathbf{V}$ is finite. Equivalently, $\mathbf{V}$ is locally finite if the relatively free lattice $\mathcal{F}_{\mathbf{V}}(n)$ is finite for every integer $n > 0$.

Theorem 8.1. *If $\mathcal{L}$ is a finite lattice and $\mathbf{V} = \mathrm{HSP}(\mathcal{L})$, then*

$$|\mathcal{F}_{\mathbf{V}}(n)| \leq |L|^{|L|^n}.$$

Hence $\mathrm{HSP}(\mathcal{L})$ *is locally finite.*

Proof. If $\mathbf{K}$ is any collection of lattices and $\mathbf{V} = \mathrm{HSP}(\mathbf{K})$, then $\mathcal{F}_{\mathbf{V}}(X) \cong \mathrm{FL}(X)/\theta$ where θ is the intersection of all homomorphism kernels $\ker f$ such that $f : \mathrm{FL}(X) \to \mathcal{L}$ for some $\mathcal{L} \in \mathbf{K}$. (This is the technical way of saying that $\mathrm{FL}(X)/\theta$ satisfies exactly the equations which hold in every member of $\mathbf{K}$.) When $\mathbf{K}$ consists of a single finite lattice $\{\mathcal{L}\}$ and $|X| = n$, then there are $|L|^n$ distinct mappings of X into L, and hence $|L|^n$ distinct homomorphisms $f_i : \mathrm{FL}(X) \to \mathcal{L}$ $(1 \leq i \leq |L|^n)$.[1] The range of each f_i is a sublattice of $\mathcal{L}$. Hence $\mathcal{F}_{\mathbf{V}}(X) \cong \mathrm{FL}(X)/\theta$ with $\theta = \bigcap \ker f_i$ means that $\mathcal{F}_{\mathbf{V}}(X)$ is a subdirect product of $|L|^n$ sublattices of $\mathcal{L}$, and so a sublattice of the direct product $\prod_{1 \leq i \leq |L|^n} \mathcal{L} = \mathcal{L}^{|L|^n}$, making its cardinality at most $|L|^{|L|^n}$.[2] $\square$

We should note that not every locally finite lattice variety is generated by a finite lattice.

Now it is clear that there is a unique minimum nontrivial lattice variety, *viz.*, the one generated by the two element lattice $\mathbf{2}$, which is isomorphic to a sublattice of any nontrivial lattice. We want to show that $\mathrm{HSP}(\mathbf{2})$ is the variety of all distributive lattices.

Lemma 8.2. *The following lattice equations are equivalent.*

(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$
(2) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$
(3) $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$

[1] The kernels of distinct homomorphisms need not be distinct, of course, but that is okay.

[2] It is sometimes useful to view this argument constructively: $\mathcal{F}_{\mathbf{V}}(X)$ is the sublattice of $\mathcal{L}^{|L|^n}$ generated by the vectors $\overline{x}$ $(x \in X)$ with $\overline{x}_i = f_i(x)$ for $1 \leq i \leq |L|^n$.

Thus each of these equations determines the variety **D** *of all distributive lattices.*

Proof. If (1) holds in a lattice $\mathcal{L}$, then for any x, y, $z \in L$ we have

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\ &= x \vee (x \wedge z) \vee (y \wedge z) \\ &= x \vee (y \wedge z)\end{aligned}$$

whence (2) holds. Thus (1) implies (2), and dually (2) implies (1).

Similarly, applying (1) to the left hand side of (3) yields the right hand side, so (1) implies (3). Conversely, assume that (3) holds in a lattice $\mathcal{L}$. For $x \geq y$, equation (3) reduces to $x \wedge (y \vee z) = y \vee (x \wedge z)$, which is the modular law, so $\mathcal{L}$ must be modular. Now for arbitrary x, y, z in $\mathcal{L}$, meet x with both sides of (3) and then use modularity to obtain

$$\begin{aligned}x \wedge (y \vee z) &= x \wedge [(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)] \\ &= (x \wedge y) \vee (x \wedge z) \vee (x \wedge y \wedge z) \\ &= (x \wedge y) \vee (x \wedge z)\end{aligned}$$

since $x \geq (x \wedge y) \vee (x \wedge z)$. Thus (3) implies (1). (Note that since (3) is self-dual, the second argument actually makes the first one redundant.) □

In the first Corollary of the next chapter, we will see that a lattice is distributive if and only if it contains neither $\mathcal{N}_5$ nor $\mathcal{M}_3$ as a sublattice. But before that, let us look at the wonderful representation theory of distributive lattices. A few moments reflection on the kernel of a homomorphism $h : \mathcal{L} \twoheadrightarrow \mathbf{2}$ should yield the following conclusions.[3]

Lemma 8.3. *Let $\mathcal{L}$ be a lattice and $h : \mathcal{L} \twoheadrightarrow \mathbf{2} = \{0, 1\}$ a surjective homomorphism. Then $h^{-1}(0)$ is an ideal of $\mathcal{L}$, $h^{-1}(1)$ is a filter, and L is the disjoint union of $h^{-1}(0)$ and $h^{-1}(1)$.*

Conversely, if I is an ideal of $\mathcal{L}$ and F a filter such that $L = I \dot{\cup} F$ (disjoint union), then the map $h : \mathcal{L} \to \mathbf{2}$ given by

$$h(x) = \begin{cases} 0 & \text{if } x \in I, \\ 1 & \text{if } x \in F. \end{cases}$$

is a surjective homomorphism.

This raises the question: *When is the complement $L - I$ of an ideal a filter?* The answer is easy. A proper ideal I of a lattice $\mathcal{L}$ is said to be *prime* if $x \wedge y \in I$ implies

[3]This is one point where we really don't want to assume that $\mathcal{L}$ has a 0 and 1. So in this chapter, an *ideal* of a lattice means a nonempty subset I such that $x \vee y \in I$ whenever x, $y \in I$, and $z \in I$ whenever $z \leq x \in I$. A *filter* is defined dually.

$x \in I$ or $y \in I$. Dually, a proper filter F is *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$. It is straightforward that the complement of an ideal I is a filter iff I is a prime ideal iff $L - I$ is a prime filter.

This simple observation allows us to work with prime ideals or prime filters (interchangeably), rather than ideal/filter pairs, and we shall do so.

Theorem 8.4. *Let $\mathcal{D}$ be a distributive lattice, and let $a \not\leq b$ in $\mathcal{D}$. Then there exists a prime filter F with $a \in F$ and $b \notin F$.*

Proof. Now $1/a$ is a filter of $\mathcal{D}$ containing a and not b, so by Zorn's Lemma there is a maximal such filter (with respect to set containment), say M. For any $x \notin M$, the filter generated by x and M must contain b, whence $b \geq x \wedge m$ for some $m \in M$. Suppose x, $y \notin M$, with say $b \geq x \wedge m$ and $b \geq y \wedge n$ where m, $n \in M$. Then by distributivity

$$b \geq (x \wedge m) \vee (y \wedge n) = (x \vee y) \wedge (x \vee n) \wedge (m \vee y) \wedge (m \vee n).$$

The last three terms are in M, so we must have $x \vee y \notin M$. Thus M is a prime filter. $\square$

Now let $\mathcal{D}$ be any distributive lattice, and let $T_{\mathcal{D}} = \{\varphi \in \mathbf{Con}\ \mathcal{D} : \mathcal{D}/\varphi \cong \mathbf{2}\}$. Theorem 8.4 says that if $a \neq b$ in $\mathcal{D}$, then there exists $\varphi \in T_{\mathcal{D}}$ with $(a, b) \notin \varphi$, whence $\bigcap T_{\mathcal{D}} = 0$ in $\mathbf{Con}\ \mathcal{D}$, i.e., $\mathcal{D}$ is a subdirect product of two element lattices.

Corollary. *The two element lattice* $\mathbf{2}$ *is the only subdirectly irreducible distributive lattice. Hence* $\mathbf{D} = \mathrm{HSP}(\mathbf{2})$.

Corollary. $\mathbf{D}$ *is locally finite.*

Another consequence of Theorem 8.4 is that every distributive lattice can be embedded into a lattice of subsets, with set union and intersection as the lattice operations.

Theorem 8.5. *Let $\mathcal{D}$ be a distributive lattice, and let S be the set of all prime filters of $\mathcal{D}$. Then the map $\phi : \mathcal{D} \to \mathfrak{P}(S)$ by*

$$\phi(x) = \{F \in S : x \in F\}$$

is a lattice embedding.

For finite distributive lattices, this representation takes on a particularly nice form. Recall that an element $p \in L$ is said to be *join prime* if it is nonzero and $p \leq x \vee y$ implies $p \leq x$ or $p \leq y$. In a finite lattice, prime filters are necessarily of the form $1/p$ where p is a join prime element.

Theorem 8.6. *Let $\mathcal{D}$ be a finite distributive lattice, and let $J(\mathcal{D})$ denote the ordered set of all nonzero join irreducible elements of $\mathcal{D}$. Then the following are true.*

(1) *Every element of $J(\mathcal{D})$ is join prime.*
(2) *$\mathcal{D}$ is isomorphic to the lattice of order ideals $\mathcal{O}(J(\mathcal{D}))$.*
(3) *Every element $a \in D$ has a unique irredundant join decomposition $a = \bigvee A$ with $A \subseteq J(\mathcal{D})$.*

Proof. In a distributive lattice, every join irreducible element is join prime, because $p \leq x \vee y$ is the same as $p = p \wedge (x \vee y) = (p \wedge x) \vee (p \wedge y)$.

For any finite lattice, the map $\phi : \mathcal{L} \to \mathcal{O}(J(\mathcal{L}))$ given by $\phi(x) = x/0 \cap J(\mathcal{L})$ is order preserving (in fact, meet preserving) and one-to-one. To establish the isomorphism of (2), we need to know that for a distributive lattice it is onto. If $\mathcal{D}$ is distributive and I is an order ideal of $J(\mathcal{D})$, then for $p \in J(\mathcal{D})$ we have by (1) that $p \leq \bigvee I$ iff $p \in I$, and hence $I = \phi(\bigvee I)$.

The join decomposition of (3) is then obtained by taking A to be the set of maximal elements of $a/0 \cap J(\mathcal{D})$. $\square$

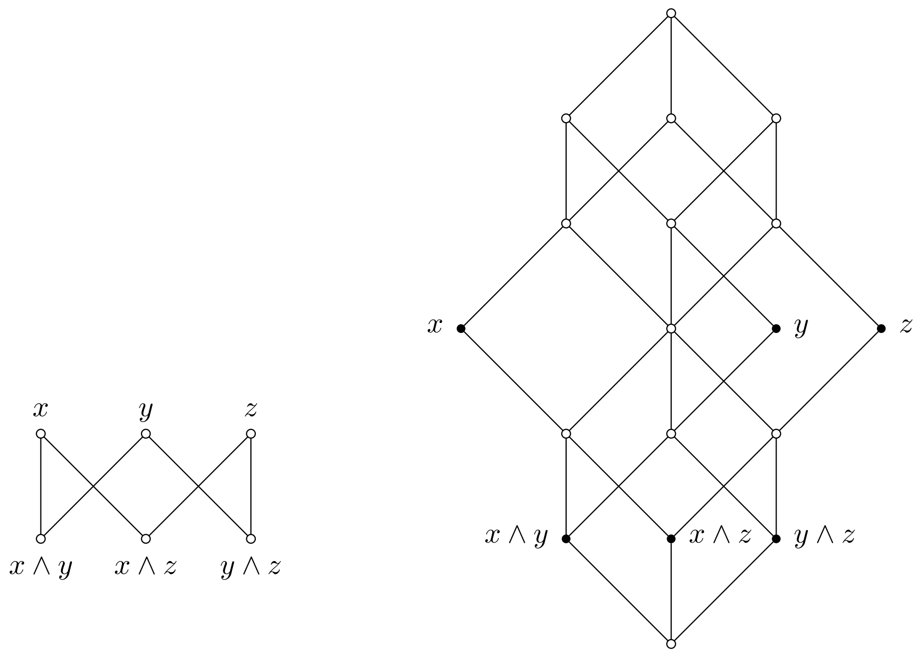
It is clear that the same proof works if $\mathcal{D}$ is an algebraic distributive lattice whose compact elements satisfy the DCC. In Theorem 10.6 we will characterize those distributive lattices isomorphic to $\mathcal{O}(\mathcal{P})$ for some ordered set $\mathcal{P}$.

As an application, we can give a neat description of the free distributive lattice $\mathcal{F}_{\mathbf{D}}(n)$ for any finite n, which we already know to be a finite distributive lattice. Let $X = \{x_1, \ldots, x_n\}$. Now it is not hard to see that any element in a free distributive lattice can be written as a join of meets of generators, $w = \bigvee w_i$ with $w_i = x_{i_1} \wedge \ldots \wedge x_{i_k}$. Another easy argument shows that the meet of a nonempty proper subset of the generators is join prime in $\mathcal{F}_{\mathbf{D}}(X)$; note that $\bigwedge \emptyset = 1$ and $\bigwedge X = 0$ do not count. (See Exercise 3). Thus the set of join irreducible elements of $\mathcal{F}_{\mathbf{D}}(X)$ is isomorphic to the (dual of, but it is self-dual) ordered set of nonempty, proper subsets of X, and the free distributive lattice is isomorphic to the lattice of order ideals of that. As an example, $\mathcal{F}_{\mathbf{D}}(3)$ and its ordered set of join irreducibles are shown in Figure 8.1.

Dedekind [6] showed that $|\mathcal{F}_{\mathbf{D}}(3)| = 18$ and $|\mathcal{F}_{\mathbf{D}}(4)| = 166$. Several other small values are known exactly, and the rest can be obtained in principle, but they grow quickly (see Quackenbush [10]). While there exist more accurate expressions, the simplest estimate is an asymptotic formula due to D. J. Kleitman:

$$\log_2 \, |\mathcal{F}_{\mathbf{D}}(n)| \sim \binom{n}{\lfloor n/2 \rfloor}.$$

The representation by sets of Theorem 8.5 does not preserve infinite joins and meets. The corresponding characterization of complete distributive lattices which have a complete representation as a lattice of subsets is derived from work of Alfred Tarski and S. Papert [9], and was surely known to both of them. An element p of a complete lattice $\mathcal{L}$ is said to be *completely join prime* if $p \leq \bigvee X$ implies $p \leq x$



$J(\mathcal{F}_\mathbf{D}(3))$ $\mathcal{F}_\mathbf{D}(3)$

FIGURE 8.1

for some $x \in X$. It is not necessary to assume that $\mathcal{D}$ is distributive in the next theorem, though of course it will turn out to be so.

Theorem 8.7. *Let $\mathcal{D}$ be a complete lattice. There exists a complete lattice embedding $\phi : \mathcal{D} \to \mathcal{P}(X)$ for some set X if and only if $x \not\leq y$ in $\mathcal{D}$ implies there exists a completely join prime element p with $p \leq x$ and $p \not\leq y$.*

Thus, for example, the interval $[0, 1]$ in the real numbers is a complete distributive lattice which cannot be represented as a complete lattice of subsets of some set.

In a lattice with 0 and 1, the pair of elements a and b are said to be *complements* if $a \wedge b = 0$ and $a \vee b = 1$. A lattice is *complemented* if every element has at least one complement. For example, the lattice of subspaces of a vector space is a complemented modular lattice. In general, an element can have many complements, but it is not hard to see that each element in a distributive lattice can have at most one complement.

A *Boolean algebra* is a complemented distributive lattice. Of course, the lattice $\mathfrak{P}(X)$ of subsets of a set is a Boolean algebra. On the other hand, it is easy to see that $\mathcal{O}(\mathcal{P})$ is complemented if and only if $\mathcal{P}$ is an antichain, in which case $\mathcal{O}(\mathcal{P}) = \mathfrak{P}(\mathcal{P})$. Thus *every finite Boolean algebra is isomorphic to the lattice $\mathfrak{P}(A)$ of subsets of its atoms.*

For a very different example, the finite and cofinite subsets of an infinite set form a Boolean algebra.

If we regard Boolean algebras as algebras $\mathcal{B} = \langle B, \wedge, \vee, 0, 1, ^c \rangle$, then they form a variety, and hence there is a *free Boolean algebra* FBA(X) generated by a set X. If X is finite, say $X = \{x_1, \ldots, x_n\}$, then FBA(X) has 2^n atoms, *viz.,* all meets $z_1 \wedge \ldots \wedge z_n$ where each z_i is either x_i or x_i^c. Thus in this case FBA$(X) \cong \mathfrak{P}(A)$ where $|A| = 2^n$. On the other hand, if X is infinite then FBA(X) has no atoms; if $|X| = \aleph_0$, then FBA(X) is the unique (up to isomorphism) countable atomless Boolean algebra!

Another natural example is the Boolean algebra of all clopen (closed and open) subsets of a topological space. In fact, by adding a topology to the representation of Theorem 8.5, we obtain the celebrated Stone representation theorem for Boolean algebras [13]. Recall that a topological space is *totally disconnected* if for every pair of distinct points x, y there is a clopen set V with $x \in V$ and $y \notin V$.

Theorem 8.8. *Every Boolean algebra is isomorphic to the Boolean algebra of clopen subsets of a compact totally disconnected (Hausdorff) space.*

Proof. Let $\mathcal{B}$ be a distributive lattice. (We will add the other properties to make $\mathcal{B}$ a Boolean algebra as we go along.) Let $\mathfrak{F}_p$ be the set of all prime filters of $\mathcal{B}$, and for $x \in B$ let

$$V_x = \{F \in \mathfrak{F}_p : x \in F\}.$$

The sets V_x will form a basis for the Stone topology on $\mathfrak{F}_p$.

With only trivial changes, the argument for Theorem 8.4 yields the following stronger version.

Sublemma A. *Let $\mathcal{B}$ be a distributive lattice, G a filter on $\mathcal{B}$ and $x \notin G$. Then there exists a prime filter $F \in \mathfrak{F}_p$ such that $G \subseteq F$ and $x \notin F$.*

Next we establish the basic properties of the sets V_x, all of which are easy to prove.

(1) $V_x \subseteq V_y$ iff $x \leq y$.
(2) $V_x \cap V_y = V_{x \wedge y}$.
(3) $V_x \cup V_y = V_{x \vee y}$.
(4) If $\mathcal{B}$ has a least element 0, then $V_0 = \emptyset$. Thus $V_x \cap V_y = \emptyset$ iff $x \wedge y = 0$.
(5) If $\mathcal{B}$ has a greatest element 1, then $V_1 = \mathfrak{F}_p$. Thus $V_x \cup V_y = \mathfrak{F}_p$ iff $x \vee y = 1$.

Property (3) is where we use the primality of the filters in the sets V_x. In particular, the family of sets V_x is closed under finite intersections, and of course $\bigcup_{x \in B} V_x = \mathfrak{F}_p$, so we can legitimately take $\{V_x : x \in B\}$ as a basis for a topology on $\mathfrak{F}_p$.

Now we would like to show that if $\mathcal{B}$ has a largest element 1, then $\mathfrak{F}_p$ is a compact space. It suffices to consider covers by basic open sets, so this follows from the next Sublemma.

Sublemma B. *If $\mathcal{B}$ has a greatest element 1 and $\bigcup_{x \in S} V_x = \mathfrak{F}_p$, then there exists a finite subset $T \subseteq S$ such that $\bigvee T = 1$, and hence $\bigcup_{x \in T} V_x = \mathfrak{F}_p$.*

Proof. Set $I_0 = \{\bigvee T : T \subseteq S,\ T \text{ finite}\}$. If $1 \notin I_0$, then I_0 generates an ideal I of $\mathcal{B}$ with $1 \notin I$. By the dual of Sublemma A, there exists a prime ideal H containing I and not 1. Its complement $B - H$ is a prime filter K. Then $K \notin \bigcup_{x \in S} V_x$, else $z \in K$ for some $z \in S$, whilst $z \in I_0 \subseteq B - K$. This contradicts our hypothesis, so we must have $1 \in I_0$, as claimed. □

The argument thus far has only required that $\mathcal{B}$ be a distributive lattice with 1. For the last two steps, we need $\mathcal{B}$ to be Boolean. Let x^c denote the complement of x in $\mathcal{B}$.

First, note that by properties (4) and (5) above, $V_x \cap V_{x^c} = \emptyset$ and $V_x \cup V_{x^c} = \mathfrak{F}_p$. Thus each set V_x ($x \in B$) is clopen. On the other hand, let W be a clopen set. As it is open, $W = \bigcup_{x \in S} V_x$ for some set $S \subseteq B$. But W is also a closed subset of the compact space $\mathfrak{F}_p$, and hence compact. Thus $W = \bigcup_{x \in T} V_x = V_{\bigvee T}$ for some finite $T \subseteq S$. Therefore W *is a clopen subset of* $\mathfrak{F}_p$ *if and only if* $W = V_x$ *for some* $x \in B$.

It remains to show that $\mathfrak{F}_p$ is totally disconnected (which makes it Hausdorff). Let F and G be distinct prime filters on $\mathcal{B}$, with say $F \not\subseteq G$. Let $x \in F - G$. Then $F \in V_x$ and $G \notin V_x$, so that V_x is a clopen set containing F and not G. □

There are similar topological representation theorems for arbitrary distributive lattices, the most useful being that due to Hilary Priestley in terms of ordered topological spaces. A good introduction is in Davey and Priestley [5].

In 1904 Huntington [8] conjectured that every uniquely complemented lattice must be distributive (and hence a Boolean algebra). It turns out that if we assume almost any additional finiteness condition on a uniquely complemented lattice, then it must be distributive. As an example, we have the following theorem of Garrett Birkhoff and Morgan Ward [4].

Theorem 8.9. *Every complete, atomic, uniquely complemented lattice is isomorphic to the Boolean algebra of all subsets of its atoms.*

Other finiteness restrictions which insure that a uniquely complemented lattice will be distributive include weak atomicity (Bandelt and Padmanabhan [3]) and upper continuity (Bandelt [2] and Saliĭ [11], [12] independently). A monograph written by Saliĭ [14] gives an excellent survey of results of this type.

Nonetheless, Huntington's conjecture is very far from true. In 1945, R. P. Dilworth [7] proved that *every lattice can be embedded in a uniquely complemented lattice.* (For a strengthened version, see Adams and Sichler [1]).

Exercises for Chapter 8

1. Show that a lattice $\mathcal{L}$ is distributive if and only if $x \vee (y \wedge z) \geq (x \vee y) \wedge z$ for all x, y, $z \in L$. (J. Bowden)

2. (a) Prove that every maximal ideal of a distributive lattice is prime.

(b) Show that a distributive lattice $\mathcal{D}$ with 0 and 1 is complemented if and only if every prime ideal of $\mathcal{D}$ is maximal.

3. These are the details of the construction of the free distributive lattice given in the text. Let X be a finite set.

(a) Let δ denote the kernel of the natural homomorphism from $\mathrm{FL}(X) \twoheadrightarrow \mathcal{F}_{\mathbf{D}}(X)$ with $x \mapsto x$. Thus $u \;\delta\; v$ iff $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ in all distributive lattices. Prove that for every $w \in \mathrm{FL}(X)$ there exists w' which is a join of meets of generators such that $w \;\delta\; w'$. (Show that the set of all such elements w is a sublattice of $\mathrm{FL}(X)$ containing the generators.)

(b) Let $\mathcal{L}$ be any lattice generated by a set X, and let $\emptyset \subset Y \subset X$. Show that for all $w \in L$, either $w \geq \bigwedge Y$ or $w \leq \bigvee (X - Y)$.

(c) Show that $\bigwedge Y \nleq \bigvee (X - Y)$ in $\mathcal{F}_{\mathbf{D}}(X)$ by exhibiting a homomorphism $h : \mathcal{F}_{\mathbf{D}}(X) \to \mathbf{2}$ with $h(\bigwedge Y) \nleq h(\bigvee (X - Y))$.

(d) Generalize these results to the case when X is a finite ordered set (as in the next exercise).

4. Find the free distributive lattice generated by

(a) $\{x_0, x_1, y_0, y_1\}$ with $x_0 < x_1$ and $y_0 < y_1$,

(b) $\{x_0, x_1, x_2, y\}$ with $x_0 < x_1 < x_2$.

5. Let $\mathcal{P} = \mathcal{Q} \dot\cup \mathcal{R}$ be the disjoint union of two ordered sets, so that q and r are incomparable whenever $q \in Q$, $r \in R$. Show that $\mathcal{O}(\mathcal{P}) \cong \mathcal{O}(\mathcal{Q}) \times \mathcal{O}(\mathcal{R})$.

6. Let $\mathcal{D}$ be a distributive lattice with 0 and 1, and let x and y be complements in $\mathcal{D}$. Prove that $\mathcal{D} \cong 1/x \times 1/y$. (Dually, $\mathcal{D} \cong x/0 \times y/0$; in fact, $1/x \cong y/0$ and $1/y \cong x/0$. This explains why $\mathbf{Con}\, \mathcal{L}_1 \times \mathcal{L}_2 \cong \mathbf{Con}\, \mathcal{L}_1 \times \mathbf{Con}\, \mathcal{L}_2$ (Exercise 5.6).)

7. Show that the following are true in a finite distributive lattice $\mathcal{D}$.

(a) For each join irreducible element x of $\mathcal{D}$, let $\kappa(x) = \bigvee \{y \in D : y \ngeq x\}$. Then $\kappa(x)$ is meet irreducible and $\kappa(x) \ngeq x$.

(b) For each $x \in J(\mathcal{D})$, $D = 1/x \,\dot\cup\, \kappa(x)/0$.

(c) The map $\kappa : J(\mathcal{D}) \to M(\mathcal{D})$ is an order isomorphism.

8. A join semilattice with 0 is *distributive* if $x \leq y \vee z$ implies there exist $y' \leq y$ and $z' \leq z$ such that $x = y' \vee z'$. Prove that an algebraic lattice is distributive if and only if its compact elements form a distributive semilattice.

9. Prove Theorem 8.7.

10. Prove Papert's characterization of lattices of closed sets of a topological space [9]: *Let $\mathcal{D}$ be a complete distributive lattice. There is a topological space $\mathcal{T}$ and an isomorphism ϕ mapping $\mathcal{D}$ onto the lattice of closed subsets of $\mathcal{T}$, preserving finite joins and infinite meets, if and only if $x \nleq y$ in $\mathcal{D}$ implies there exists a (finitely) join prime element p with $p \leq x$ and $p \nleq y$.*

References

1. M. E. Adams and J. Sichler, *Lattices with unique complementation*, Pacific J. Math. **92** (1981), 1–13.
2. H. J. Bandelt, *Complemented continuous lattices*, Arch. Math. (Basel) **36** (1981), 474–475.
3. H. J. Bandelt and R. Padmanabhan, *A note on lattices with unique comparable complements*, Abh. Math. Sem. Univ. Hamburg **48** (1979), 112–113.

4. G. Birkhoff and M. Ward, *A characterization of Boolean algebras*, Ann. of Math. **40** (1939), 609–610.
5. B. Davey and H. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 1990.
6. R. Dedekind, *Über die drei Moduln erzeugte Dualgruppe*, Math. Annalen **53** (1900), 371–403.
7. R. P. Dilworth, *Lattices with unique complements*, Trans. Amer. Math. Soc. **57** (1945), 123–154.
8. E. V. Huntington, *Sets of independent postulates for the algebra of logic*, Trans. Amer. Math. Soc. **5** (1904), 288–309.
9. S. Papert, *Which distributive lattices are lattices of closed sets?*, Proc. Cambridge Phil. Soc. **55** (1959), 172–176.
10. R. Quackenbush, *Dedekind's problem*, Order **2** (1986), 415–417.
11. V. N. Saliĭ, *A compactly generated lattice with unique complements is distributive*, Mat. Zametki **12** (1972), 617–620. (Russian)
12. V. N. Saliĭ, *A continuous uniquely complemented lattice is distributive*, Fifth All-Union Conf. Math. Logic, Abstracts of Reports, Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1979, p. 134. (Russian)
13. M. H. Stone, *The theory of representations of Boolean Algebras*, Trans. Amer. Math. Soc. **40** (1936), 37–111.
14. V. N. Saliĭ, *Lattices with unique complements*, Translations of the Amer. Math. Soc., vol. 69, Amer. Math. Soc., Providence, R. I., 1988.

9. Modular Lattices

To dance beneath the diamond sky with one hand waving free ...
–Bob Dylan

The modular law was invented by Dedekind to reflect a crucial property of the lattice of subgroups of an abelian group, or more generally the lattice of normal subgroups of a group. In this chapter on modular lattices you will see the lattice theoretic versions of some familiar theorems from group theory. This will lead us naturally to consider semimodular lattices.

Likewise, the lattice of submodules of a module over a ring is modular. Thus our results on modular lattices apply to the lattice of ideals of a ring, or the lattice of subspaces of a vector space. These applications make modular lattices particularly important.

The smallest nonmodular lattice is $\mathcal{N}_5$, which is called the *pentagon*. Dedekind's characterization of modular lattices is simple [3].

Theorem 9.1. *A lattice is modular if and only if it does not contain the pentagon as a sublattice.*

Proof. Clearly, a modular lattice cannot contain $\mathcal{N}_5$ as a sublattice. Conversely, suppose $\mathcal{L}$ is a nonmodular lattice. Then there exist $x > y$ and z in $\mathcal{L}$ such that $x \wedge (y \vee z) > y \vee (x \wedge z)$. Now the lattice freely generated by x, y, z with $x \geq y$ is shown in Figure 9.1; you should verify that it is correct. The elements $x \wedge (y \vee z)$, $y \vee (x \wedge z)$, z, $x \wedge z$ and $y \vee z$ form a pentagon there, and likewise in $\mathcal{L}$. Since the pentagon is subdirectly irreducible and $x \wedge (y \vee z)/y \vee (x \wedge z)$ is the critical quotient, these five elements are distinct. $\square$

Birkhoff [1] showed that there is a similar characterization of distributive lattices within the class of modular lattices. The *diamond* is $\mathcal{M}_3$, which is the smallest nondistributive modular lattice.

Theorem 9.2. *A modular lattice is distributive if and only if it does not contain the diamond as a sublattice.*

Proof. Again clearly, a distributive lattice cannot have a sublattice isomorphic to $\mathcal{M}_3$. Conversely, let $\mathcal{L}$ be a nondistributive modular lattice. Then, by Lemma 8.2, there exist x, y, z in $\mathcal{L}$ such that $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) > (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. Now the free modular lattice $\mathcal{F}_{\mathbf{M}}(3)$ is diagrammed in Figure 9.2; again you should

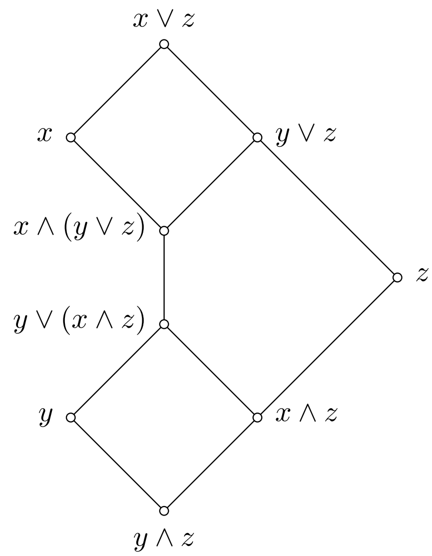


FIGURE 9.1: $\mathrm{FL}(\mathbf{2}+\mathbf{1})$

verify that it is correct.[1] The interval between the two elements above is a diamond in $\mathcal{F}_{\mathbf{M}}(3)$, and the corresponding elements will form a diamond in $\mathcal{L}$.

The details go as follows. The middle elements of our diamond should be

$$[x \wedge (y \vee z)] \vee (y \wedge z) = [x \vee (y \wedge z)] \wedge (y \vee z)$$
$$[y \wedge (x \vee z)] \vee (x \wedge z) = [y \vee (x \wedge z)] \wedge (x \vee z)$$
$$[z \wedge (x \vee y)] \vee (x \wedge y) = [z \vee (x \wedge y)] \wedge (x \vee y)$$

where in each case the equality follows from modularity. The join of the first pair of elements is (using the first expressions)

$$\begin{aligned}[x \wedge (y \vee z)] \vee (y \wedge z) \vee [y \wedge (x \vee z)] \vee (x \vee z) &= [x \wedge (y \vee z)] \vee [y \wedge (x \vee z)] \\ &= [(x \wedge (y \vee z)) \vee y] \wedge (x \vee z) \\ &= (x \vee y) \wedge (x \vee z) \wedge (y \vee z).\end{aligned}$$

Symmetrically, the other pairs of elements also join to $(x \vee y) \wedge (x \vee z) \wedge (y \vee z)$. Since the second expression for each element is dual to the first, each pair of these three elements meets to $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. Because the diamond is simple, the five elements will be distinct, and hence form a sublattice isomorphic to $\mathcal{M}_3$. □

[1]Recall from Chapter 7, though, that $\mathcal{F}_{\mathbf{M}}(n)$ is infinite and has an unsolvable word problem for $n \geq 4$.

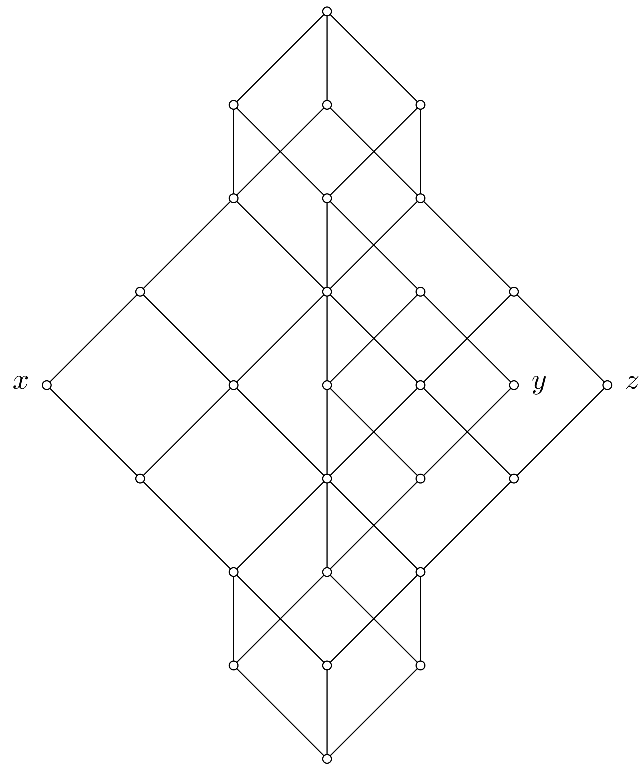


FIGURE 9.2: $\mathcal{F}_{\mathbf{M}}(3)$

Corollary. *A lattice is distributive if and only if it has neither* $\mathcal{N}_5$ *nor* $\mathcal{M}_3$ *as a sublattice.*

The preceding two results tell us something more about the bottom of the lattice Λ of lattice varieties. We already know that the trivial variety $\mathbf{T}$ is uniquely covered by $\mathbf{D} = \mathrm{HSP}(\mathbf{2})$, which is in turn covered by $\mathrm{HSP}(\mathcal{N}_5)$ and $\mathrm{HSP}(\mathcal{M}_3)$. By the Corollary, these are the only two varieties covering $\mathbf{D}$.

Much more is known about the bottom of Λ. Both $\mathrm{HSP}(\mathcal{N}_5)$ and $\mathrm{HSP}(\mathcal{M}_3)$ are covered by their join $\mathrm{HSP}\{\mathcal{N}_5, \mathcal{M}_3\} = \mathrm{HSP}(\mathcal{N}_5 \times \mathcal{M}_3)$. George Grätzer and Bjarni Jónsson ([6], [7]) showed that $\mathrm{HSP}(\mathcal{M}_3)$ has two additional covers, and Jónsson and Ivan Rival [8] proved that $\mathrm{HSP}(\mathcal{N}_5)$ has exactly fifteen other covers, each generated by a finite subdirectly irreducible lattice. You are encouraged to try and find these covers. Because of Jónsson's Lemma, it is never hard to tell if $\mathrm{HSP}(\mathcal{K})$ covers $\mathrm{HSP}(\mathcal{L})$ when $\mathcal{K}$ and $\mathcal{L}$ are finite lattices; the hard part is determining whether your list of covers is complete. Since a variety generated by a finite lattice can have infinitely many covering varieties, or a covering variety generated by an infinite subdirectly irreducible lattice, this can only be done near the bottom of Λ.

Now we return to modular lattices. For any two elements a, b in a lattice $\mathcal{L}$ there are natural maps $\mu_a : (a \vee b)/b \to a/(a \wedge b)$ and $\nu_b : a/(a \wedge b) \to (a \vee b)/b$ given by

$$\mu_a(x) = x \wedge a$$
$$\nu_b(x) = x \vee b.$$

Dedekind showed that these maps play a special role in the structure of modular lattices.

Theorem 9.3. *If a and b are elements of a modular lattice $\mathcal{L}$, then μ_a and ν_b are mutually inverse isomorphisms, whence $(a \vee b)/b \cong a/(a \wedge b)$.*

Proof. Clearly, μ_a and ν_b are order preserving. They are mutually inverse maps by modularity: for if $x \in (a \vee b)/b$, then

$$\nu_b \mu_a(x) = b \vee (a \wedge x) = (b \vee a) \wedge x = x$$

and, dually, $\mu_a \nu_b(y) = y$ for all $y \in a/(a \wedge b)$. □

Corollary. *In a modular lattice, $a \succ a \wedge b$ if and only if $a \vee b \succ b$.*

For groups we actually have somewhat more. The First Isomorphism Theorem says that if $\mathcal{A}$ and $\mathcal{B}$ are subgroups of a group $\mathcal{G}$, and $\mathcal{B}$ is normal in $\mathcal{A} \vee \mathcal{B}$, then the quotient groups $\mathcal{A}/\mathcal{A} \wedge \mathcal{B}$ and $\mathcal{A} \vee \mathcal{B}/\mathcal{B}$ are isomorphic.

A lattice $\mathcal{L}$ is said to be *semimodular* (or *upper semimodular*) if $a \succ a \wedge b$ implies $a \vee b \succ b$ in $\mathcal{L}$. Equivalently, $\mathcal{L}$ is semimodular if $u \succ v$ implies $u \vee x \succeq v \vee x$, where $a \succeq b$ means a covers or equals b. The dual property is called *lower semimodular*. Traditionally, *semimodular* by itself always refers to upper semimodularity. Clearly the Corollary shows that modular lattices are both upper and lower semimodular. A strongly atomic, algebraic lattice which is both upper and lower semimodular is modular. (See Theorem 3.7 of [2]; you are asked to prove the finite dimensional version of this in Exercise 3.)

Our next result is a version of the Jordan-Hölder Theorem for semimodular lattices, first proved for modular lattices by Dedekind in 1897.

Theorem 9.4. *Let $\mathcal{L}$ be a semimodular lattice and let $a < b$ in $\mathcal{L}$. If there is a finite maximal chain from a to b, then every chain from a to b is finite, and all the maximal ones have the same length.*

Proof. We are given that there is a finite maximal chain in b/a, say

$$a = a_0 \prec a_1 \prec \cdots \prec a_n = b.$$

If $n = 1$, i.e., $a \prec b$, then the theorem is trivially true. So we may assume inductively that it holds for any interval containing a maximal chain of length less than n.

Let C be another maximal chain in b/a. If, perchance, $c \geq a_1$ for all $c \in C - \{a\}$, then $C - \{a\}$ is a maximal chain in b/a_1. In that case, $C - \{a\}$ has length $n - 1$ by induction, and so C has length n.

Thus we may assume that there is an element $d \in C - \{a\}$ such that $d \not\geq a_1$. Moreover, since b/a_1 has finite length, we can choose d such that $d \vee a_1$ is minimal, i.e., $e \vee a_1 \geq d \vee a_1$ for all $e \in C - \{a\}$. We can show that $d \succ a$ as follows. Suppose not. Then $d > e > a$ for some $e \in L$; since C is a maximal chain containing a and d, we can choose $e \in C$. Now $a_1 \succ a = d \wedge a_1 = e \wedge a_1$. Hence by semimodularity $d \vee a_1 \succ d$ and $e \vee a_1 \succ e$. But the choice of d implies $e \vee a_1 \geq d \vee a_1 \succ d > e$, contradicting the second covering relation. Therefore $d \succ a$.

Now we are quickly done. As a_1 and d both cover a, their join $a_1 \vee d$ covers both of them. Since $a_1 \vee d \succ a_1$, every maximal chain in $b/(a_1 \vee d)$ has length $n - 2$. Then every chain in b/d has length $n - 1$, and C has length n, as desired. □

Now let $\mathcal{L}$ be a semimodular lattice in which every principal ideal $x/0$ has a finite maximal chain. Then we can define a *dimension function* δ on $\mathcal{L}$ by letting $\delta(x)$ be the length of a maximal chain from 0 to x:

$$\delta(x) = n \quad \text{if} \quad 0 = c_0 \prec c_1 \prec \cdots \prec c_n = x.$$

By Theorem 9.4, δ is well defined. For semimodular lattices the properties of the dimension function can be summarized as follows.

Theorem 9.5. *If $\mathcal{L}$ is a semimodular lattice and every principal ideal has only finite maximal chains, then the dimension function on $\mathcal{L}$ has the following properties.*

(1) $\delta(0) = 0$,
(2) $x > y$ *implies* $\delta(x) > \delta(y)$,
(3) $x \succ y$ *implies* $\delta(x) = \delta(y) + 1$,
(4) $\delta(x \vee y) + \delta(x \wedge y) \leq \delta(x) + \delta(y)$.

Conversely, if $\mathcal{L}$ is a lattice which admits an integer valued function δ satisfying (1)–(4), then $\mathcal{L}$ is semimodular and principal ideals have only finite maximal chains.

Proof. Given a semimodular lattice $\mathcal{L}$ in which principal ideals have only finite maximal chains, properties (1) and (2) are obvious, while (3) is a consequence of Theorem 9.4. The only (not very) hard part is to establish the inequality (4). Let x and y be elements of $\mathcal{L}$, and consider the join map $\nu_x : y/(x \wedge y) \to (x \vee y)/x$ defined by $\nu_x(z) = z \vee x$. Recall that, by semimodularity, $u \succ v$ implies $u \vee x \succeq v \vee x$. Hence ν_x takes maximal chains in $y/(x \wedge y)$ to maximal chains in $(x \vee y)/x$. So the length of $(x \vee y)/x$ is at most that of $y/(x \wedge y)$, i.e.,

$$\delta(x \vee y) - \delta(x) \leq \delta(y) - \delta(x \wedge y)$$

which establishes the desired inequality.

Conversely, suppose $\mathcal{L}$ is a lattice which admits a function δ satisfying (1)–(4). Note that, by (2), $\delta(x) \geq \delta(z) + 2$ whenever $x > y > z$; hence $\delta(x) = \delta(z) + 1$ implies $x \succ z$.

To establish semimodularity, assume $a \succ a \wedge b$ in $\mathcal{L}$. By (3) we have $\delta(a) = \delta(a \wedge b) + 1$, and so by (4)

$$\begin{aligned} \delta(a \vee b) + \delta(a \wedge b) &\leq \delta(a) + \delta(b) \\ &= \delta(a \wedge b) + 1 + \delta(b) \end{aligned}$$

whence $\delta(a \vee b) \leq \delta(b) + 1$. As $a \vee b > b$, in fact $\delta(a \vee b) = \delta(b) + 1$ and $a \vee b \succ b$, as desired.

For any $a \in L$, if $a = a_k > a_{k-1} > \cdots > a_0$ is any chain in $a/0$, then $\delta(a_j) > \delta(a_{j-1}$ so $k \leq \delta(a)$. Thus every chain in $a/0$ has length at most $\delta(a)$. □

For modular lattices, the map μ_x is an isomorphism, so we obtain instead equality. It also turns out that we can dispense with the third condition, though this is not very important.

Theorem 9.6. *If $\mathcal{L}$ is a modular lattice and every principal ideal has only finite maximal chains, then*

(1) $\delta(0) = 0$,
(2) $x > y$ *implies* $\delta(x) > \delta(y)$,
(3) $\delta(x \vee y) + \delta(x \wedge y) = \delta(x) + \delta(y)$.

Conversely, if $\mathcal{L}$ is a lattice which admits an integer-valued function δ satisfying (1)–(3), then $\mathcal{L}$ is modular and principal ideals have only finite maximal chains.

At this point, it is perhaps useful to have some examples of semimodular lattices. The lattice of equivalence relations **Eq** X is semimodular, but nonmodular for $|X| \geq 4$. The lattice in Figure 9.3 is semimodular, but not modular. We will see more semimodular lattices as we go along, arising from group theory (subnormal subgroups) in this chapter and from geometry in Chapter 11.

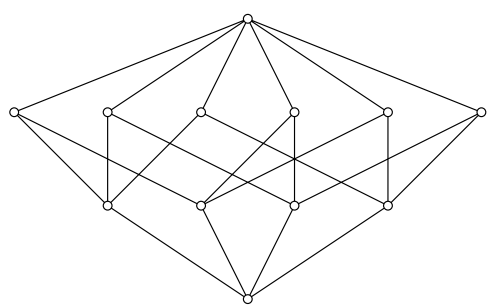


FIGURE 9.3

For our applications to group theory, we need a supplement to Theorem 9.4. This in turn requires a definition. We say that a quotient a/b *transposes up* to

c/d if $a \vee d = c$ and $a \wedge d = b$. We then say that c/d *transposes down* to a/b. We then define *projectivity* to be the smallest equivalence relation on the set of all quotients of a lattice $\mathcal{L}$ which contains all transposed pairs $\langle x/(x \wedge y), (x \vee y)/y \rangle$. Thus a/b is projective to c/d if and only if there exists a sequence of quotients $a/b = a_0/b_0, a_1/b_1, \ldots, a_n/b_n = c/d$ such that a_i/b_i and a_{i+1}/b_{i+1} are transposes (up or down).

The strengthened version of Theorem 9.4, again due to Dedekind for modular lattices, goes thusly.

Theorem 9.7. *Let C and D be two maximal chains in a finite length semimodular lattice, say*

$$0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1$$
$$0 = d_0 \prec d_1 \prec \cdots \prec d_n = 1.$$

Then there is a permutation π of the set $\{1, \ldots, n\}$ such that c_i/c_{i-1} is projective to $d_{\pi(i)}/d_{\pi(i)-1}$ for all i.

Proof. Again we use induction on n. We may assume $c_1 \neq d_1$, for otherwise the result follows by induction. Then $c_1/0$ transposes up to $(c_1 \vee d_1)/d_1$, and $d_1/0$ transposes up to $(c_1 \vee d_1)/c_1$.

Let $c_1 \vee d_1 = e_2 \prec e_3 \prec \cdots \prec e_n = 1$ be a maximal chain in $1/(c_1 \vee d_1)$. By induction, there is a permutation σ of $\{2, \ldots, n\}$ such that c_i/c_{i-1} is projective to $e_{\sigma(i)}/e_{\sigma(i)-1}$ if $\sigma(i) \neq 2$, and c_i/c_{i-1} is projective to $e_2/c_1 = (c_1 \vee d_1)/c_1$ if $\sigma(i) = 2$. Similarly, there is a permutation τ of $\{2, \ldots, n\}$ such that d_j/d_{j-1} is projective to $e_{\tau(j)}/e_{\tau(j)-1}$ if $\tau(j) \neq 2$, and d_j/d_{j-1} is projective to $e_2/d_1 = (c_1 \vee d_1)/d_1$ if $\tau(j) = 2$. Now just check that the permutation π of $\{1, \ldots, n\}$ given by

$$\pi(k) = \begin{cases} \tau^{-1}\sigma(k) & \text{if } k > 1 \text{ and } \sigma(k) \neq 2 \\ 1 & \text{if } k > 1 \text{ and } \sigma(k) = 2 \\ \tau^{-1}(2) & \text{if } k = 1 \end{cases}$$

has the property that c_k/c_{k-1} is projective to $d_{\pi(k)}/d_{\pi(k)-1}$.

This argument is illustrated in Figure 9.4. □

Theorems 9.4 and 9.7 are important in group theory. A *chief series* of a group $\mathcal{G}$ is a maximal chain in the lattice of normal subgroups $\mathcal{N}(\mathcal{G})$. Since $\mathcal{N}(\mathcal{G})$ is modular, our theorems apply.

Corollary. *If a group $\mathcal{G}$ has a finite chief series of length k,*

$$\{1\} = N_0 < N_1 < \cdots < N_k = \mathcal{G}$$

then every chief series of $\mathcal{G}$ has length k. Moreover, if

$$\{1\} = H_0 < H_1 < \cdots < H_k = \mathcal{G}$$

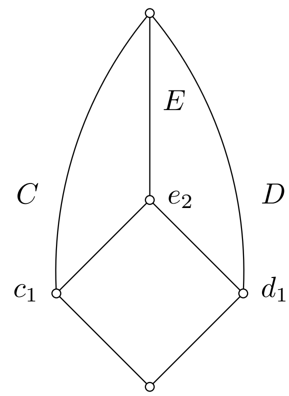


Figure 9.4

is another chief series of $\mathcal{G}$, then there is a permutation π of $\{1, \ldots, k\}$ such that $H_i/H_{i-1} \cong N_{\pi(i)}/N_{\pi(i)-1}$ for all i.

A subgroup H is *subnormal* in a group $\mathcal{G}$, written $H \lhd\lhd \mathcal{G}$, if there is a chain in **Sub** $\mathcal{G}$,

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_k = \mathcal{G}$$

with each H_{i-1} normal in H_i (but not necessarily in $\mathcal{G}$ for $i < k$). Herman Wielandt proved that the subnormal subgroups of a finite group form a lattice [10].

Theorem 9.8. *If $\mathcal{G}$ is a finite group, then the subnormal subgroups of $\mathcal{G}$ form a lower semimodular sublattice $\mathcal{SN}(\mathcal{G})$ of* **Sub** $\mathcal{G}$.

Proof. Let H and K be subnormal in $\mathcal{G}$, with say

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_m = \mathcal{G}$$
$$K = K_0 \lhd K_1 \lhd \ldots \lhd K_n = \mathcal{G}.$$

Then $H \cap K_i \lhd H \cap K_{i+1}$, and so we have the series

$$H \cap K \lhd H \cap K_1 \lhd H \cap K_2 \lhd \ldots H \cap \mathcal{G} = H \lhd H_1 \lhd \ldots \lhd \mathcal{G}.$$

Thus $H \cap K \lhd\lhd \mathcal{G}$. Note that this argument shows that *if H, $K \lhd\lhd \mathcal{G}$ and $K \leq H$, then $K \lhd\lhd H$.*

The proof that $\mathcal{SN}(\mathcal{G})$ is closed under joins is a bit trickier. Let H, $K \lhd\lhd \mathcal{G}$ as before. Without loss of generality, H and K are incomparable. By induction, we may assume that $|\mathcal{G}|$ is minimal and that the result holds for larger subnormal subgroups of $\mathcal{G}$, i.e.,

(1) the join of subnormal subgroups is again subnormal in any group $\mathcal{G}'$ with $|\mathcal{G}'| < |\mathcal{G}|$,
(2) if $H < L \lhd\lhd \mathcal{G}$, then $L \vee K \lhd\lhd \mathcal{G}$; likewise, if $K < M \lhd\lhd \mathcal{G}$, then $H \vee M \lhd\lhd \mathcal{G}$.

If there is a subnormal proper subgroup S of $\mathcal{G}$ which contains both H and K, then H and K are subnormal subgroups of S (by the observation above). In that case, $H \vee K \lhd\!\lhd S$ by the first assumption, whence $H \vee K \lhd\!\lhd \mathcal{G}$. Thus we may assume that

(3) no subnormal proper subgroup of $\mathcal{G}$ contains both H and K.

Combining this with assumption (2) yields

(4) $H_1 \vee K = \mathcal{G} = H \vee K_1$.

Finally, if both H and K are normal in $\mathcal{G}$, then so is $H \vee K$. Thus we may assume (by symmetry) that

(5) H is not normal in $\mathcal{G}$, and hence $H < H_1 \leq H_{m-1} < \mathcal{G}$.

Now $\mathcal{G}$ is generated by the set union $H_1 \cup K$ (assumption (4)), so we must have $x^{-1}Hx \neq H$ for some $x \in H_1 \cup K$. But $H \lhd H_1$, so $k^{-1}Hk \neq H$ for some $k \in K$.

However, $k^{-1}Hk$ is a subnormal subgroup of H_{m-1}, because

$$k^{-1}Hk \lhd k^{-1}H_1k \lhd \ldots \lhd k^{-1}H_{m-1}k = H_{m-1}$$

as $H_{m-1} \lhd \mathcal{G}$. Thus, by assumption (1), $H \vee k^{-1}Hk$ is a subnormal subgroup of H_{m-1}, and hence of $\mathcal{G}$. But $H < H \vee k^{-1}Hk \leq H \vee K$, so $(H \vee k^{-1}Hk) \vee K = H \vee K$. Therefore $H \vee K$ is subnormal in $\mathcal{G}$ by assumption (2).

Finally, if $H \vee K \succ H$ in $\mathcal{SN}(\mathcal{G})$, then $H \lhd H \vee K$ (using the observation after the first argument), and $(H \vee K)/H$ is simple. By the First Isomorphism Theorem, $K/(H \wedge K)$ is likewise simple, so $K \succ H \wedge K$. Thus $\mathcal{SN}(\mathcal{G})$ is lower semimodular. □

A maximal chain in $\mathcal{SN}(\mathcal{G})$ is called a *composition series* for $\mathcal{G}$. As $\mathcal{SN}(\mathcal{G})$ is lower semimodular, the duals of Theorems 9.4 and 9.7 yield the following important structural theorem for groups.

Corollary. *If a finite group $\mathcal{G}$ has a composition series of length n,*

$$\{1\} = H_0 \lhd H_1 \lhd \ldots \lhd H_n = \mathcal{G}$$

then every composition series of $\mathcal{G}$ has length n. Moreover, if

$$\{1\} = K_0 \lhd K_1 \lhd \ldots \lhd K_n = \mathcal{G}$$

is another composition series for $\mathcal{G}$, then there is a permutation π of $\{1, \ldots, n\}$ such that $K_i/K_{i-1} \cong H_{\pi(i)}/H_{\pi(i)-1}$ for all i.

A *finite decomposition* of an element $a \in L$ is an expression $a = \bigwedge Q$ where Q is a finite set of meet irreducible elements. If $\mathcal{L}$ satisfies the ACC, then every element has a finite decomposition. We have seen that every element of a finite distributive lattice has a unique irredundant decomposition. In a finite dimensional modular lattice, an element can have many different finite decompositions, but the number of elements in any irredundant decomposition is always the same. This is a consequence of the following replacement property (known as the Kurosh-Ore Theorem).

Theorem 9.9. *If a is an element of a modular lattice and*

$$a = q_1 \wedge \ldots \wedge q_m = r_1 \wedge \ldots \wedge r_n$$

are two irredundant decompositions of a, then $m = n$ and for each q_i there is an r_j such that

$$a = r_j \wedge \bigwedge_{k \neq i} q_k$$

is an irredundant decomposition.

Proof. Let $a = \bigwedge Q = \bigwedge R$ be two irredundant finite decompositions (dropping the subscripts temporarily). Fix $q \in Q$, and let $\overline{q} = \bigwedge(Q - \{q\})$. By modularity, $q \vee \overline{q}/q \cong \overline{q}/q \wedge \overline{q} = \overline{q}/a$. Since q is meet irreducible in $\mathbf{L}$, this implies that a is meet irreducible in $\overline{q}/a$. However, $a = \overline{q} \wedge \bigwedge R = \bigwedge_{r \in R}(\overline{q} \wedge r)$ takes place in $\overline{q}/a$, so we must have $a = \overline{q} \wedge r$ for some $r \in R$.

Next we observe that $a = r \wedge \bigwedge(Q - \{q\})$ is irredundant. For if not, we would have $a = r \wedge \bigwedge S$ irredundantly for some proper subset $S \subset Q - \{q\}$. Reapplying the first argument to the two decompositions $a = r \wedge \bigwedge S = \bigwedge Q$ with the element r, we obtain $a = q' \wedge \bigwedge S$ for some $q' \in Q$, contrary to the irredundance of Q.

It remains to show that $|Q| = |R|$. Let $Q = \{q_1, \ldots, q_m\}$ say. By the first part, there is an element $r_1 \in R$ such that $a = r_1 \wedge \bigwedge(Q - \{q_1\}) = \bigwedge R$ irredundantly. Applying the argument to these two decompositions and q_2, there is an element $r_2 \in R$ such that $a = r_1 \wedge r_2 \wedge \bigwedge(Q - \{q_1, q_2\}) = \bigwedge R$. Moreover, r_1 and r_2 are distinct, for otherwise we would have $a = r_1 \wedge \bigwedge(Q - \{q_1, q_2\})$, contradicting the irredundance of $a = r_1 \wedge \bigwedge(Q - \{q_1\})$. Continuing, we can replace q_3 by an element r_3 of R, distinct from r_1 and r_2, and so forth. After m steps, we obtain $a = r_1 \wedge \cdots \wedge r_m$, whence $R = \{r_1, \ldots, r_m\}$. Thus $|Q| = |R|$. □

With a bit of effort, this can be improved to a *simultaneous* exchange theorem.

Theorem 9.10. *If a is an element of a modular lattice and $a = \bigwedge Q = \bigwedge R$ are two irredundant finite decompositions of a, then for each $q \in Q$ there is an $r \in R$ such that*

$$a = r \wedge \bigwedge(Q - \{q\}) = q \wedge \bigwedge(R - \{r\}).$$

The proof of this, and much more on the general theory of decompositions in lattices, can be found in Crawley and Dilworth [2]; see also Dilworth [5].

Now Theorems 9.9 and 9.10 are exactly what we want in a finite dimensional modular lattice. However, in algebraic modular lattices, finite decompositions into meet irreducible elements need not coincide with the (possibly infinite) decomposition into completely meet irreducible elements given by Birkhoff's Theorem. Consider, for example, the chain $\mathcal{C} = (\omega + 1)^d$, the dual of $\omega + 1$. This satisfies the ACC, and hence is algebraic. The least element of $\mathcal{C}$ is meet irreducible, but not completely meet irreducible. In the direct product $\mathcal{C}^n$, the least element has a finite decomposition into n meet irreducible elements, but every decomposition into completely meet irreducibles is necessarily infinite.

Fortunately, the proof of Theorem 9.9 adapts nicely to give us a version suitable for algebraic modular lattices.

Theorem 9.11. *Let a be an element of a modular lattice. If $a = \bigwedge Q$ is a finite, irredundant decomposition into completely meet irreducible elements, and $a = \bigwedge R$ is another decomposition into meet irreducible elements, then there exists a finite subset $R' \subseteq R$ with $|R'| = |Q|$ such that $a = \bigwedge R'$ irredundantly.*

The application of Theorem 9.11 to subdirect products is immediate.

Corollary. *Let $\mathcal{A}$ be an algebra such that* **Con** *$\mathcal{A}$ is a modular lattice. If $\mathcal{A}$ has a finite subdirect decomposition into subdirectly irreducible algebras, then every irredundant subdirect decomposition of $\mathcal{A}$ into subdirectly irreducible algebras has the same number of factors.*

A more important application is to the theory of direct decompositions of congruence modular algebras. (The corresponding congruences form a complemented sublattice of **Con** $\mathcal{A}$.) This subject is treated thoroughly in McKenzie, McNulty and Taylor [9].

Let us close this section by mentioning a nice combinatorial result about finite modular lattices, due to R. P. Dilworth [4].

Theorem 9.12. *In a finite modular lattice $\mathcal{L}$, let $J_k(\mathcal{L})$ be the set of elements which cover exactly k elements, and let $M_k(\mathcal{L})$ be the set of elements which are covered by exactly k elements. Then $|J_k(\mathcal{L})| = |M_k(\mathcal{L})|$ for any integer $k \geq 0$.*

In particular, the number of join irreducible elements in a finite modular lattice is equal to the number of meet irreducible elements.

We will return to modular lattices in Chapter 12.

Exercises for Chapter 9

1. (a) Prove that a lattice $\mathcal{L}$ is distributive if and only if it has the property that $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$ imply $a = b$.

(b) Show that $\mathcal{L}$ is modular if and only if, whenever $a \geq b$ and $c \in L$, $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$ imply $a = b$.

2. Show that every finite dimensional distributive lattice is finite.

3. Prove that if a finite dimensional lattice is both upper and lower semimodular, then it is modular.

4. Prove that the following conditions are equivalent for a strongly atomic, algebraic lattice.

(i) $\mathcal{L}$ is semimodular: $a \succ a \wedge b$ implies $a \vee b \succ b$.
(ii) If a and b both cover $a \wedge b$, then $a \vee b$ covers both a and b.

(iii) If b and c are incomparable and $b \wedge c < a < c$, then there exists x such that $b \wedge c < x \leq b$ and $a = c \wedge (a \vee x)$.

5. (a) Find infinitely many simple modular lattices of width 4.

(b) Prove that the variety generated by all lattices of width ≤ 4 contains subdirectly irreducible lattices of width ≤ 4 only.

6. Prove that every arguesian lattice is modular.

7. Let $\mathcal{L}$ be a lattice, and suppose there exist an ideal I and a filter F of $\mathcal{L}$ such that $L = I \cup F$ and $I \cap F \neq \emptyset$.

(a) Show that $\mathcal{L}$ is distributive if and only if both I and F are distributive.

(b) Show that $\mathcal{L}$ is modular if and only if both I and F are modular.

(R. P. Dilworth)

8. Show that modular lattices satisfy the equation

$$x \wedge (y \vee (z \wedge (x \vee t))) = x \wedge (z \vee (y \wedge (x \vee t))).$$

9. Let C and D be two chains in a modular lattice $\mathcal{L}$. Prove that $C \cup D$ generates a distributive sublattice of $\mathcal{L}$. (Bjarni Jónsson)

10. Let a and b be two elements in a modular lattice $\mathcal{L}$ such that $a \wedge b = 0$. Prove that the sublattice generated by $a/0 \cup b/0$ is isomorphic to the direct product $a/0 \times b/0$.

11. Prove Theorem 9.11. (Mimic the proof of Theorem 9.9.)

12. Let $\mathcal{A} = \prod_{i \in \omega} \mathbb{Z}_2$ be the direct product of countably many copies of the two element group. Describe two decompositions of 0 in **Sub** $\mathcal{A}$, say $0 = \bigwedge Q = \bigwedge R$, such that $|Q| = \aleph_0$ and $|R| = 2^{\aleph_0}$.

References

1. G. Birkhoff, *Lattice Theory*, First edition, Colloquium Publications, vol. 25, Amer. Math. Soc., Providence, R. I., 1940.
2. P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, N. J., 1973.
3. R. Dedekind, *Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler*, Festschrift der Herzogl. technische Hochschule zur Naturforscher-Versammlung, Braunschweig (1897), Reprinted in "Gesammelte mathematische Werke", Vol. 2, pp. 103–148, Chelsea, New York, 1968.
4. R. P. Dilworth, *Proof of a conjecture on finite modular lattices*, Ann. of Math. **60** (1954), 359–364.
5. R. P. Dilworth, *Structure and Decomposition Theory*, Proceedings of Symposia on Pure Mathematics: Lattice Theory, vol. II, Amer. Math. Soc., Providence, R. I., 1961.
6. G. Grätzer, *Equational classes of lattices*, Duke Jour. Math. **33** (1966), 613–622.
7. B. Jónsson, *Equational classes of lattices*, Math. Scand. **22** (1968), 187–196.
8. B. Jónsson and I. Rival, *Lattice varieties covering the smallest non-modular variety*, Pacific J. Math. **82** (1979), 463–478.
9. R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties*, vol. I, Wadsworth and Brooks-Cole, Belmont, CA, 1987.
10. H. Wielandt, *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Zeit. **45** (1939), 209–244.

10. Finite Lattices and their Congruence Lattices

If memories are all I sing
I'd rather drive a truck.
–Ricky Nelson

In this chapter we want to study the structure of finite lattices, and how it is reflected in their congruence lattices. There are different ways of looking at lattices, each with its own advantages. For the purpose of studying congruences, it is useful to represent a finite lattice as the lattice of closed sets of a closure operator on its set of join irreducible elements. This is an efficient way to encode the structure, and will serve us well.[1]

The approach to congruences taken in this chapter is not the traditional one. It evolved from techniques developed over a period of time by Ralph McKenzie, Bjarni Jónsson, Alan Day, Ralph Freese and J. B. Nation for dealing with various specific questions (see [1], [4], [6], [7], [8], [9]). Gradually, the general usefulness of these methods dawned on us.

In the simplest case, recall that a finite distributive lattice $\mathcal{L}$ is isomorphic to the lattice of order ideals $\mathcal{O}(J(\mathcal{L}))$, where $J(\mathcal{L})$ is the ordered set of nonzero join irreducible elements of $\mathcal{L}$. This reflects the fact that join irreducible elements in a distributive lattice are join prime. In a nondistributive lattice, we seek a modification which will keep track of the ways in which one join irreducible is below the join of others. In order to do this, we must first develop some terminology.

Rather than just considering finite lattices, we can include with modest additional effort a larger class of lattices satisfying a strong finiteness condition. Recall that a lattice $\mathcal{L}$ is *principally chain finite* if no principal ideal of $\mathcal{L}$ contains an infinite chain (equivalently, every principal ideal $x/0$ satisfies the ACC and DCC). In Theorem 11.1, we will see where this class arises naturally in an important setting.[2]

Recall that if $X, Y \subseteq L$, we say that X *refines* Y (written $X \ll Y$) if for each $x \in X$ there exists $y \in Y$ with $x \leq y$. It is easy to see that the relation $\ll$ is a quasiorder (reflexive and transitive), but not in general antisymmetric. Note $X \subseteq Y$ implies $X \ll Y$.

If $q \in J(\mathcal{L})$ is completely join irreducible, let q_* denote the unique element of L with $q \succ q_*$. Note that if $\mathcal{L}$ is principally chain finite, then q_* exists for each $q \in J(\mathcal{L})$.

[1] For an alternate approach, see Appendix. 3

[2] Many of the results in this chapter can be generalized to arbitrary lattices. However, these generalizations have not yet proved to be very useful unless one assumes at least the DCC.

A *join expression* of $a \in L$ is a finite set B such that $a = \bigvee B$. A join expression $a = \bigvee B$ is *minimal* if it is irredundant and B cannot be properly refined, i.e., $B \subseteq J(\mathcal{L})$ and $a > b_* \vee \bigvee(B - \{b\})$ for each $b \in B$. An equivalent way to write this technically is that $a = \bigvee B$ minimally if $a = \bigvee C$ and $C \ll B$ implies $B \subseteq C$.

A *join cover* of $p \in L$ is a finite set A such that $p \leq \bigvee A$. A join cover A of p is *minimal* if $\bigvee A$ is irredundant and A cannot be properly refined to another join cover of p, i.e., $p \leq \bigvee B$ and $B \ll A$ implies $A \subseteq B$.

We define a binary relation $\underline{D}$ on $J(\mathcal{L})$ as follows: $p \; \underline{D} \; q$ if there exists $x \in L$ such that $p \leq q \vee x$ but $p \not\leq q_* \vee x$. This relation will play an important role in our analysis of the congruences of a principally chain finite lattice.[3]

The following lemma summarizes some properties of principally chain finite lattices and the relation $\underline{D}$.

Lemma 10.1. *Let $\mathcal{L}$ be a principally chain finite lattice.*

(1) *If $b \not\leq a$ in $\mathcal{L}$, then there exists $p \in J(\mathcal{L})$ with $p \leq b$ and $p \not\leq a$.*
(2) *Every join expression in $\mathcal{L}$ refines to a minimal join expression, and every join cover refines to a minimal join cover.*
(3) *For $p, q \in J(\mathcal{L})$ we have $p \; \underline{D} \; q$ if and only if $q \in A$ for some minimal join cover A of p.*

Proof. (1) Since $b \not\leq a$ and $b/0$ satisfies the DCC, the set $\{x \in b/0 : x \not\leq a\}$ has at least one minimal element p. Because $y < p$ implies $y \leq a$ for any $y \in L$, we have $\bigvee\{y \in L : y < p\} \leq p \wedge a < p$, and hence $p \in J(\mathcal{L})$ with $p_* = p \wedge a$.

(2) Suppose $\mathcal{L}$ contains an element s with a join representation $s = \bigvee F$ which does not refine to a minimal one. Since the DCC holds in $s/0$, there is an element $t \leq s$ minimal with respect to having a join representation $t = \bigvee A$ which fails to refine to a minimal one. Clearly t is join reducible, and there is a proper, irredundant join expression $t = \bigvee B$ with $B \ll A$.

Let $B = \{b_1, \ldots, b_k\}$. Using the DCC on $b_1/0$, we can find $c_1 \leq b_1$ such that $t = c_1 \vee b_2 \vee \ldots \vee b_k$, but c_1 cannot be replaced by any lower element: $t > u \vee b_2 \vee \ldots b_k$ whenever $u < c_1$. Now apply the same argument to b_2 and $\{c_1, b_2, \ldots, b_k\}$. After k such steps we obtain a join cover C which refines B and is minimal *pointwise*: no element can be replaced by a (single) lower element.

The elements of C may not be join irreducible, but each element of C is strictly below t, and hence has a minimal join expression. Choose a minimal join expression E_c for each $c \in C$. It is not hard to check that $E = \bigcup_{c \in C} E_c$ is a minimal join expression for t, and $E \ll C \ll B \ll A$, which contradicts the choice of t and B.

Now let $u \in L$ and let A be a join cover of u, i.e., $u \leq \bigvee A$. We can find $B \subseteq A$ such that $u \leq \bigvee B$ irredundantly. As above, refine B to a pointwise minimal

[3]Note that $\underline{D}$ is reflexive, i.e., $p \; \underline{D} \; p$ for all $p \in J(\mathbf{L})$. The relation D, defined similarly except that it requires $p \neq q$, is also important, and $\underline{D}$ stands for "D or equal to." For describing congruences, it makes more sense to use $\underline{D}$ rather than D.

join cover C. Now we know that minimal join expressions exist, so we may define $E = \bigcup_{c \in C} E_c$ exactly as before. Then E will be a minimal join cover of u, and again $E \ll C \ll B \ll A$.

(3) Assume $p \,\underline{D}\, q$, and let $x \in L$ be such that $p \leq q \vee x$ but $p \nleq q_* \vee x$. By (2), we can find a minimal join cover A of p with $A \ll \{q, x\}$. Since $p \nleq q_* \vee x$, we must have $q \in A$.

Conversely, if A is a minimal join cover of p, and $q \in A$, then we fulfill the definition of $p \,\underline{D}\, q$ by setting $x = \bigvee (A - \{q\})$. $\square$

Now we want to define a closure operator on the join irreducible elements of a principally chain finite lattice. This closure operator should encode the structure of $\mathcal{L}$ in the same way the order ideal operator $\mathcal{O}$ does for a finite distributive lattice. For $S \subseteq J(\mathcal{L})$, let

$$\Gamma(S) = \{p \in J(\mathcal{L}) : p \leq \bigvee F \text{ for some finite } F \subseteq S\}.$$

It is easy to check that Γ is an algebraic closure operator. The compact (i.e., finitely generated) Γ-closed sets are of the form $\Gamma(F) = \{p \in J(\mathcal{L}) : p \leq \bigvee F\}$ for some finite subset F of $J(\mathcal{L})$. In general, we would expect these to be only a join subsemilattice of the lattice $\mathcal{C}_\Gamma$ of closed sets; however, for a principally chain finite lattice $\mathcal{L}$ the compact closed sets actually form an ideal (and hence a sublattice) of $\mathcal{C}_\Gamma$. For if $S \subseteq \Gamma(F)$ with F finite, then $S \subseteq \bigvee F/0$, which satisfies the ACC. Hence $\{\bigvee G : G \subseteq S \text{ and } G \text{ is finite}\}$ has a largest element. So $\bigvee S = \bigvee G$ for some finite $G \subseteq S$, from which it follows that $\Gamma(S) = \Gamma(G)$, and $\Gamma(S)$ is compact. In particular, if $\mathcal{L}$ has a largest element 1, then every closed set will be compact.

With that preliminary observation out of the way, we proceed with our generalization of the order ideal representation for finite distributive lattices.

Theorem 10.2. *If $\mathcal{L}$ is a principally chain finite lattice, then the map ϕ with $\phi(x) = \{p \in J(\mathcal{L}) : p \leq x\}$ is an isomorphism of $\mathcal{L}$ onto the lattice of compact Γ-closed subsets of $J(\mathcal{L})$.*

Proof. Note that if $x = \bigvee A$ is a minimal join expression, then $\phi(x) = \Gamma(A)$, so $\phi(x)$ is indeed a compact Γ-closed set. The map ϕ is clearly order preserving, and it is one-to-one by part (1) of Lemma 10.1. Finally, ϕ is onto because $\Gamma(F) = \phi(\bigvee F)$ for each finite $F \subseteq J(\mathcal{L})$. $\square$

To use this result, we need a structural characterization of Γ-closed sets.

Theorem 10.3. *Let $\mathcal{L}$ be a principally chain finite lattice. A subset C of $J(\mathcal{L})$ is Γ-closed if and only if*

(1) *C is an order ideal of $J(\mathcal{L})$, and*
(2) *if A is a minimal join cover of $p \in J(\mathcal{L})$ and $A \subseteq C$, then $p \in C$.*

Proof. It is easy to see that Γ-closed sets have these properties. Conversely, let $C \subseteq J(\mathcal{L})$ satisfy (1) and (2). We want to show $\Gamma(C) \subseteq C$. If $p \in \Gamma(C)$, then $p \leq \bigvee F$ for some finite subset $F \subseteq C$. By Lemma 10.1(2), there is a minimal join cover A of p refining F; since C is an order ideal, $A \subseteq C$. But then the second closure property gives that $p \in C$, as desired. $\square$

In words, Theorem 10.3 says that for principally chain finite lattices, Γ is determined by the order on $J(\mathcal{L})$ and the minimal join covers of elements of $J(\mathcal{L})$. Hence, by Theorem 10.2, $\mathcal{L}$ is determined by the same factors. Now we would like to see how much of this information we can extract from **Con** $\mathcal{L}$. The answer is, "not much." We will see that from **Con** $\mathcal{L}$ we can find $J(\mathcal{L})$ modulo a certain equivalence relation. We can determine nothing of the order on $J(\mathcal{L})$, nor can we recover the minimal join covers, but we can recover the $\underline{D}$ relation (up to the equivalence). This turns out to be enough to characterize the congruence lattices of principally chain finite lattices.

Now for a group $\mathcal{G}$, the map $\tau : \mathbf{Con}\,\mathcal{G} \to \mathcal{N}(\mathcal{G})$ given by $\tau(\theta) = \{x \in G : x\,\theta\,1\}$ is a lattice isomorphism. The next two theorems and corollary establish a similar correspondence for principally chain finite lattices.

Theorem 10.4. *Let $\mathcal{L}$ be a principally chain finite lattice. Let σ map* **Con** $\mathcal{L}$ *to the lattice of subsets $\mathcal{P}(J(\mathcal{L}))$ by*

$$\sigma(\theta) = \{p \in J(\mathcal{L}) : p\;\theta\;p_*\}.$$

Then σ is a one-to-one complete lattice homomorphism.

Proof. Clearly σ is order preserving: $\theta \leq \psi$ implies $\sigma(\theta) \subseteq \sigma(\psi)$.

To see that σ is one-to-one, assume $\theta \nleq \psi$. Then there exists a pair of elements $a, b \in L$ with $a < b$ and $(a, b) \in \theta - \psi$. Since $(a, b) \notin \psi$, we also have $(x, b) \notin \psi$ for any element x with $x \leq a$. Let $p \leq b$ be minimal with respect to the property $p\,\psi\,x$ implies $x \nleq a$. We claim that p is join irreducible. If $y_1, \dots, y_n < p$, then for each i there exists an x_i such that $y_i\,\psi\,x_i \leq a$. Hence $\bigvee y_i\,\psi\,\bigvee x_i \leq a$, so $\bigvee y_i < p$. Now $p = p \wedge b\,\theta\,p \wedge a \leq p_*$, implying $p\,\theta\,p_*$, i.e., $p \in \sigma(\theta)$. But $(p, p_*) \notin \psi$ because $p_*\,\psi\,x \leq a$ for some x; thus $p \notin \sigma(\psi)$. Therefore $\sigma(\theta) \nsubseteq \sigma(\psi)$.

It is easy to see that $\sigma(\bigwedge \theta_i) = \bigcap \sigma(\theta_i)$ for any collection of congruences θ_i $(i \in I)$. Since σ is order preserving, we have $\bigcup \sigma(\theta_i) \subseteq \sigma(\bigvee \theta_i)$, and it remains to show that $\sigma(\bigvee \theta_i) \subseteq \bigcup \sigma(\theta_i)$.

If $(p, p_*) \in \bigvee \theta_i$, then there exists a connecting sequence

$$p = x_0\,\theta_{i_1}\,x_1\,\theta_{i_2}\,x_2 \dots x_{k-1}\,\theta_{i_k}\,x_k = p_*.$$

Let $y_j = (x_j \vee p_*) \wedge p$. Then $y_0 = p$, $y_k = p_*$, and $p_* \leq y_j \leq p$ implies $y_j \in \{p_*, p\}$ for each j. Moreover, we have $y_{j-1}\theta_{i_j}y_j$ for $j \geq 1$. There must exist a j with $y_{j-1} = p$ and $y_j = p_*$, whence $p\,\theta_{i_j}\,p_*$ and $p \in \sigma(\theta_{i_j}) \subseteq \bigcup \sigma(\theta_i)$. We conclude that σ also preserves arbitrary joins. $\square$

Next we need to identify the range of σ.

Theorem 10.5. *Let $\mathcal{L}$ be a principally chain finite lattice, and let $S \subseteq J(\mathcal{L})$. Then $S = \sigma(\theta)$ for some $\theta \in \mathbf{Con}\,\mathcal{L}$ if and only if $p \,\underline{D}\, q \in S$ implies $p \in S$.*

Proof. Let $S = \sigma(\theta)$. If $q \in S$ and $p \,\underline{D}\, q$, then $q \;\theta\; q_*$, and for some $x \in L$ we have $p \leq q \vee x$ but $p \nleq q_* \vee x$. Thus

$$p = p \wedge (q \vee x) \;\theta\; p \wedge (q_* \vee x) < p.$$

Hence $p \;\theta\; p_*$ and $p \in \sigma(\theta) = S$.

Conversely, assume we are given $S \subseteq J(\mathcal{L})$ satisfying the condition of the theorem. Then we must produce a congruence relation θ such that $\sigma(\theta) = S$. Let $T = J(\mathcal{L}) - S$, and note that T has the property that $q \in T$ whenever $p \,\underline{D}\, q$ and $p \in T$. Define

$$x \;\theta\; y \text{ if } x/0 \cap T = y/0 \cap T.$$

The motivation for this definition is outlined in the exercises: θ is the kernel of the *standard homomorphism* from $\mathcal{L}$ onto the join subsemilattice of $\mathcal{L}$ generated by $T \cup \{0\}$.

Three things should be clear: θ is an equivalence relation; $x \;\theta\; y$ implies $x \wedge z \;\theta\; y \wedge z$; and for $p \in J(\mathcal{L})$, $p \;\theta\; p_*$ if and only if $p \notin T$, i.e., $p \in S$. (The last statement will imply that $\sigma(\theta) = S$.) It remains to show that θ respects joins.

Assume $x \;\theta\; y$, and let $z \in L$. We want to show $(x \vee z)/0 \cap T \subseteq (y \vee z)/0 \cap T$, so let $p \in T$ and $p \leq x \vee z$. Then there exists a minimal join cover Q of p with $Q \ll \{x, z\}$. If $q \in Q$ and $q \leq z$, then of course $q \leq y \vee z$. Otherwise $q \leq x$, and since $p \in T$ and $p \,\underline{D}\, q$ (by Lemma 10.1(3)), we have $q \in T$. Thus $q \in x/0 \cap T = y/0 \cap T$, so $q \leq y \leq y \vee z$. It follows that $p \leq \bigvee Q \leq y \vee z$. This shows $(x \vee z)/0 \cap T \subseteq (y \vee z)/0 \cap T$; by symmetry, they are equal. Hence $x \vee z \;\theta\; y \vee z$. $\square$

In order to interpret the consequences of these two theorems, let $\trianglelefteq$ denote the transitive closure of $\underline{D}$ on $J(\mathcal{L})$. Then $\trianglelefteq$ is a quasiorder (reflexive and transitive), and so it induces an equivalence relation $\equiv$ on $J(\mathcal{L})$, modulo which $\trianglelefteq$ is a partial order, *viz.*, $p \equiv q$ if and only if $p \trianglelefteq q$ and $q \trianglelefteq p$. If we let $Q_{\mathcal{L}}$ denote the partially ordered set $(J(\mathcal{L})/\equiv, \trianglelefteq)$, then Theorem 10.5 translates as follows.

Corollary. *If $\mathcal{L}$ is a principally chain finite lattice, then $\mathbf{Con}\,\mathcal{L} \cong \mathcal{O}(Q_{\mathcal{L}})$.*

Because the $\underline{D}$ relation is easy to determine, it is not hard to find $Q_{\mathcal{L}}$ for a finite lattice $\mathcal{L}$. Hence this result provides a reasonably efficient algorithm for determining the congruence lattice of a finite lattice. Hopefully, the exercises will convince you of this. As an application, we have the following observation.

Corollary. *A principally chain finite lattice $\mathcal{L}$ is subdirectly irreducible if and only if $Q_{\mathcal{L}}$ has a least element.*

Now let us turn our attention to the problem of representing a given distributive algebraic lattice $\mathcal{D}$ as the congruence lattice of a lattice.[4] Not every distributive

[4]Recall from Chapter 5 that it is an open problem whether every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice.

algebraic lattice is isomorphic to $\mathcal{O}(\mathcal{P})$ for an ordered set $\mathcal{P}$. Indeed, those which are have a nice characterization.

Lemma 10.6. *The following are equivalent for a distributive algebraic lattice $\mathcal{D}$.*

(1) *$\mathcal{D}$ is isomorphic to the lattice of order ideals of an ordered set.*
(2) *Every element of $\mathcal{D}$ is a join of completely join prime elements.*
(3) *Every compact element of $\mathcal{D}$ is a join of (finitely many) join irreducible compact elements.*

Proof. An order ideal I is compact in $\mathcal{O}(\mathcal{P})$ if and only if it is finitely generated, i.e., $I = p_1/0 \cup \cdots \cup p_k/0$ for some $p_1, \ldots, p_k \in P$. Moreover, each $p_i/0$ is join irreducible in $\mathcal{O}(\mathcal{P})$. Thus $\mathcal{O}(\mathcal{P})$ has the property (3).

Note that if $\mathcal{D}$ is a distributive algebraic lattice and p is a join irreducible compact element, then p is completely join prime. For if $p \leq \bigvee U$, then $p \leq \bigvee U'$ for some finite subset $U' \subseteq U$; as join irreducible elements are join prime in a distributive lattice, this implies $p \leq u$ for some $u \in U'$. On the other hand, a completely join prime element is clearly compact and join irreducible, so these elements coincide. If every compact element is a join of join irreducible compact elements, then so is every element of $\mathcal{D}$, whence (3) implies (2).

Now assume that the completely join prime elements of $\mathcal{D}$ are join dense, and let $\mathcal{P}$ denote the set of completely join prime elements with the order they inherit from $\mathcal{D}$. Then it is straightforward to show that the map $\phi : \mathcal{D} \to \mathcal{O}(\mathcal{P})$ given by $\phi(x) = x/0 \cap P$ is an isomorphism. □

Now it is not hard to find lattices where these conditions fail. Nonetheless, distributive algebraic lattices with the properties of Lemma 10.6 are a nice class (including all finite distributive lattices), and it behooves us to try to represent each of them as **Con** $\mathcal{L}$ for some principally chain finite lattice $\mathcal{L}$. We need to begin by seeing how $Q_{\mathcal{L}}$ can be recovered from **Con** $\mathcal{L}$.

Theorem 10.7. *Let $\mathcal{L}$ be a principally chain finite lattice. A congruence relation θ is join irreducible and compact in* **Con** $\mathcal{L}$ *if and only if $\theta = \operatorname{con}(p, p_*)$ for some $p \in J$. Moreover, for $p, q \in J$, we have $\operatorname{con}(q, q_*) \leq \operatorname{con}(p, p_*)$ iff $q \trianglelefteq p$.*

Proof. We want to use the representation **Con** $\mathcal{L} \cong \mathcal{O}(Q_{\mathcal{L}})$. Note that if Q is a partially ordered set and I is an order ideal of Q, then $I = \bigcup_{x \in I} x/0$, and, of course, set union is the join operation in $\mathcal{O}(Q)$. Hence join irreducible compact ideals are exactly those of the form $x/0$ for some $x \in Q$.

Applying these remarks to our situation, using the isomorphism, join irreducible compact congruences are precisely those with $\sigma(\theta) = \{q \in J(\mathcal{L}) : q \trianglelefteq p\}$ for some $p \in J(\mathcal{L})$. Recalling that $p \in \sigma(\theta)$ if and only if $p \;\theta\; p_*$, and $\operatorname{con}(p, p_*)$ is the least congruence with $p \;\theta\; p_*$, the conclusions of the theorem follow. □

Theorem 10.8. *Let $\mathcal{D}$ be a distributive algebraic lattice which is isomorphic to $\mathcal{O}(\mathcal{P})$ for some ordered set $\mathcal{P}$. Then there is a principally chain finite lattice $\mathcal{L}$ such that $\mathcal{D} \cong$ **Con** $\mathcal{L}$.*

Proof. We must construct $\mathcal{L}$ with $Q_{\mathcal{L}} \cong \mathcal{P}$. In view of Theorem 10.3 we should try to describe $\mathcal{L}$ as the lattice of finitely generated closed sets of a closure operator on an ordered set J. Let P^0 and P^1 be two unordered copies of the base set P of $\mathcal{P}$, disjoint except on the maximal elements of $\mathcal{P}$. Thus $J = P^0 \cup P^1$ is an antichain, and $p^0 = p^1$ if and only if p is maximal in $\mathcal{P}$. Define a subset C of J to be *closed* if $\{p^j, q^k\} \subseteq C$ implies $p^i \in C$ whenever $p < q$ in $\mathcal{P}$ and $\{i, j\} = \{0, 1\}$. Our lattice $\mathcal{L}$ will consist of all finite closed subsets of J, ordered by set inclusion.

It should be clear that we have made the elements of J atoms of $\mathcal{L}$ and

$$p^i \leq p^j \vee q^k$$

whenever $p < q$ in $\mathcal{P}$. Thus $p^i \, \underline{D} \, q^k$ iff $p \leq q$. (This is where you want only one copy of each maximal element). It remains to check that $\mathcal{L}$ is indeed a principally chain finite lattice with $Q_{\mathcal{L}} \cong \mathcal{P}$, as desired. The crucial observation is that the closure of a finite set is finite. We will leave this verification to the reader. □

Theorem 10.8 is due to R. P. Dilworth in the 1940's, but his proof was never published. The construction given is from George Grätzer and E. T. Schmidt [5].

We close this section with a new look at a pair of classic results. A lattice is said to be *relatively complemented* if $a < x < b$ implies there exists y such that $x \wedge y = a$ and $x \vee y = b$.[5]

Theorem 10.9. *If $\mathcal{L}$ is a principally chain finite lattice which is either modular or relatively complemented, then the relation $\underline{D}$ is symmetric on $J(\mathcal{L})$, and hence* **Con** $\mathcal{L}$ *is a Boolean algebra.*

Proof. First assume $\mathcal{L}$ is modular, and let $p \, \underline{D} \, q$ with $p \leq q \vee x$ but $p \not\leq q_* \vee x$. Using modularity, we have

$$(q \wedge (p \vee x)) \vee x = (q \vee x) \wedge (p \vee x) \geq p,$$

so $q \leq p \vee x$. On the other hand, if $q \leq p_* \vee x$, we would have

$$p = p \wedge (q \vee x) \leq p \wedge (p_* \vee x) = p_* \vee (x \wedge p) = p_*,$$

a contradiction. Hence $q \not\leq p_* \vee x$, and $q \, \underline{D} \, p$.

Now assume $\mathcal{L}$ is relatively complemented and $p \, \underline{D} \, q$ as above. Observe that a join irreducible element in a relatively complemented lattice must be an atom.

[5] Thus a relatively complemented lattice with 0 and 1 is complemented, but otherwise it need not be.

Hence $p_* = q_* = 0$, and given x such that $p \leq q \vee x$, $p \not\leq x$, we want to find y such that $q \leq p \vee y$, $q \not\leq y$. Using the ACC in $q \vee x/0$, let y be maximal such that $x \leq y < q \vee x$ and $p \not\leq y$. If $y < p \vee y < q \vee x$, then the relative complement z of $p \vee y$ in $q \vee x/y$ satisfies $z > y$ and $z \not\geq p$, contrary to the maximality of y. Hence $p \vee y = q \vee x$, i.e., $q \leq p \vee y$. Thus $q \underline{D} p$.

Finally, if $\underline{D}$ is symmetric, then $Q_{\mathcal{L}}$ is an antichain, and thus $\mathcal{O}(Q_{\mathcal{L}})$ is isomorphic to the Boolean algebra $\mathcal{P}(Q_{\mathcal{L}})$. $\square$

A lattice is *simple* if $|L| > 1$ and $\mathcal{L}$ has no proper nontrivial congruence relations, i.e., **Con** $\mathcal{L} \cong \mathbf{2}$. Theorem 10.9 says that a subdirectly irreducible, modular or relatively complemented, principally chain finite lattice must be simple.

In the relatively complemented case we get even more. Let $\mathcal{L}_i$ $(i \in I)$ be a collection of lattices with 0. The *direct sum* $\sum \mathcal{L}_i$ is the sublattice of the direct product consisting of all elements which are only finitely non-zero. Combining Theorems 10.2 and 10.9, we obtain relatively easily a fine result of Dilworth [2].

Theorem 10.10. *A relatively complemented principally chain finite lattice is a direct sum of simple (relatively complemented principally chain finite) lattices.*

Proof. Let $\mathcal{L}$ be a relatively complemented principally chain finite lattice. Then every element of L is a finite join of join irreducible elements, every join irreducible element is an atom, and the $\underline{D}$ relation is symmetric, i.e., $p \underline{D} q$ implies $p \equiv q$. We can write $J(\mathcal{L})$ as a disjoint union of $\equiv$-classes, $J(\mathcal{L}) = \dot{\bigcup}_{i \in I} A_i$. Let

$$L_i = \{x \in L : x = \bigvee F \text{ for some finite } F \subseteq A_i\}.$$

We want to show that the L_i's are ideals (and hence sublattices) of $\mathcal{L}$, and that $\mathcal{L} \cong \sum_{i \in I} \mathcal{L}_i$.

The crucial technical detail is this: *if $p \in J(\mathcal{L})$, $F \subseteq J(\mathcal{L})$ is finite, and $p \leq \bigvee F$, then $p \equiv f$ for some $f \in F$.* For F can be refined to a minimal join cover G of p, and since join irreducible elements are atoms, we must have $G \subseteq F$. But $p \underline{D} g$ (and hence $p \equiv g$) for each $g \in G$.

Now we can show that each $\mathcal{L}_i$ is an ideal of $\mathcal{L}$. Suppose $y \leq x \in L_i$. Then $x = \bigvee F$ for some $F \subseteq A_i$, and $y = \bigvee H$ for some $H \subseteq J(\mathcal{L})$. By the preceding observation, $H \subseteq A_i$, and thus $y \in L_i$.

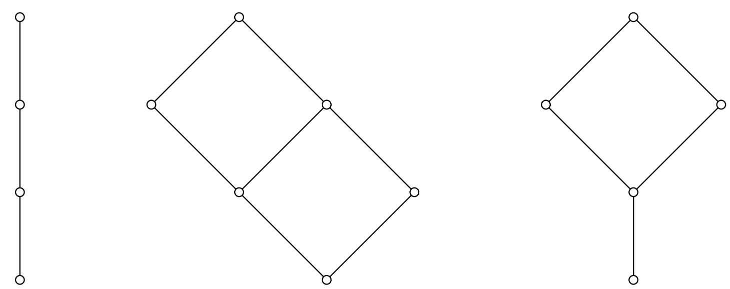
Define a map $\phi : \mathcal{L} \to \sum_{i \in I} \mathcal{L}_i$ by $\phi(x) = (x_i)_{i \in I}$, where $x_i = \bigvee (x/0 \cap A_i)$. There are several things to check: that $\phi(x)$ is only finitely nonzero, that ϕ is one-to-one and onto, and that it preserves meets and joins. None is very hard, so we will only do the last one, and leave the rest to the reader.

We want to show that ϕ preserves joins, i.e., that $(x \vee y)_i = x_i \vee y_i$. It suffices to show that if $p \in J(\mathcal{L})$ and $p \leq (x \vee y)_i$, then $p \leq x_i \vee y_i$. Since $\mathcal{L}_i$ is an ideal, we have $p \in A_i$. Furthermore, since $p \leq x \vee y$, there is a minimal join cover F of p refining $\{x, y\}$. For each $f \in F$, we have $f \leq x$ or $f \leq y$, and $p \underline{D} f$ implies $f \in A_i$; hence $f \leq x_i$ or $f \leq y_i$. Thus $p \leq \bigvee F \leq x_i \vee y_i$. $\square$

Exercises for Chapter 10

1. Do Exercise 1 of Chapter 5 using the methods of this chapter.

2. Use the construction from the proof of Theorem 10.8 to represent the distributive lattices in Figure 10.1 as congruence lattices of lattices.



(a) (b) (c)

Figure 10.1

3. Let $a = \bigvee B$ be a join expression in a lattice $\mathcal{L}$. Prove that the following two properties (used to define minimality) really are equivalent.

(a) $B \subseteq J(\mathcal{L})$ and $a > b_* \vee \bigvee(B - \{b\})$ for each $b \in B$.

(b) $a = \bigvee C$ and $C \ll B$ implies $B \subseteq C$.

4. Let $\mathcal{P}$ be an ordered set satisfying the DCC, and let $\mathcal{Q}$ be the set of finite antichains of $\mathcal{P}$, ordered by $\ll$. Show that $\mathcal{Q}$ satisfies the DCC. (This argument is rather tricky, but it is the proper explanation of Lemma 10.1(2).)

5. Let p be a join irreducible element in a principally chain finite lattice. Show that p is join prime if and only if $p \, \underline{D} \, q$ implies $p = q$.

6. Let $\mathcal{L}$ be a principally chain finite lattice, and $p \in J(\mathcal{L})$. Prove that there is a congruence ψ_p on $\mathcal{L}$ such that, for all $\theta \in \mathbf{Con}\,\mathcal{L}$, $(p, p_*) \notin \theta$ if and only if $\theta \leq \psi_p$.

(More generally, the following is true: Given a lattice $\mathcal{L}$ and a filter F of $\mathcal{L}$, there is a unique congruence ψ_F maximal with respect to the property that $(x, f) \in \theta$ implies $x \in F$ for all $x \in L$ and $f \in F$.)

7. Prove that a distributive lattice is isomorphic to $\mathcal{O}(\mathcal{P})$ for some ordered set $\mathcal{P}$ if and only if it is algebraic and dually algebraic. (This extends Lemma 10.6.)

8. Let $\mathcal{L}$ be a principally chain finite lattice, and let $T \subseteq J(\mathcal{L})$ have the property that $p \, \underline{D} \, q$ and $p \in T$ implies $q \in T$.

(a) Show that the join subsemilattice $\mathcal{S}$ of $\mathcal{L}$ generated by $T \cup \{0\}$, i.e., the set of all $\bigvee F$ where F is a finite subset of $T \cup \{0\}$, is a lattice. ($\mathcal{S}$ need not a be sublattice of $\mathcal{L}$, because the meet operation is different.)

(b) Prove that the map $f : \mathcal{L} \twoheadrightarrow \mathcal{S}$ given by $f(x) = \bigvee(x/0 \cap T)$ is a lattice homomorphism.

(c) Show that the kernel of f is the congruence relation θ in the proof of Theorem 10.5.

9. Prove that if $\mathcal{L}$ is a finite lattice, then $\mathcal{L}$ can be embedded into a finite lattice $\mathcal{K}$ such that $\mathbf{Con}\,\mathcal{L} \cong \mathbf{Con}\,\mathcal{K}$ and every element of $\mathcal{K}$ is a join of atoms. (Michael Tischendorf)

10. Express the lattice of all finite subsets of a set X as a direct sum of two-element lattices.

11. Show that if $\mathcal{A}$ is a torsion abelian group, then the compact subgroups of $\mathcal{A}$ form a principally chain finite lattice (Khalib Benabdallah).

The main arguments in this chapter originated in a slightly different setting, geared towards application to lattice varieties [7], the structure of finitely generated free lattices [4], or finitely presented lattices [3]. The last three exercises give the version of these results which has proved most useful for these types of applications, with an example.

A lattice homomorphism $f : \mathcal{L} \to \mathcal{K}$ is *lower bounded* if for every $a \in K$, the set $\{x \in L : f(x) \geq a\}$ is either empty or has a least element, which is denoted $\beta(a)$. If f is onto, this is equivalent to saying that each congruence class of $\ker f$ has a least element. For example, if $\mathcal{L}$ satisfies the DCC, then every homomorphism $f : \mathcal{L} \to \mathcal{K}$ will be lower bounded. The dual condition is called *upper bounded*. These notions were introduced by Ralph McKenzie in [7].

12. Let $\mathcal{L}$ be a lattice with 0, $\mathcal{K}$ a finite lattice, and $f : \mathcal{L} \twoheadrightarrow \mathcal{K}$ a lower bounded, surjective homomorphism. Let $T = \{\beta(p) : p \in J(\mathcal{K})\}$. Show that:

(a) $T \subseteq J(\mathcal{L})$;

(b) $\mathcal{K}$ is isomorphic to the join subsemilattice $\mathcal{S}$ of $\mathcal{L}$ generated by $T \cup \{0\}$;

(c) for each $t \in T$, every join cover of t in $\mathcal{L}$ refines to a join cover of t contained in T.

13. Conversely, let $\mathcal{L}$ be a lattice with 0, and let T be a finite subset of $J(\mathcal{L})$ satisfying condition (c) of Exercise 12. Let $\mathcal{S}$ denote the join subsemilattice of $\mathcal{L}$ generated by $T \cup \{0\}$. Prove that the map $f : \mathcal{L} \twoheadrightarrow \mathcal{S}$ given by $f(x) = \bigvee(x/0 \cap T)$ is a lower bounded lattice homomorphism with $\beta f(t) = t$ for all $t \in T$.

14. Let f be the (essentially unique) homomorphism from $FL(3)$ onto $\mathcal{N}_5$. Show that f is lower bounded. (By duality, f is also upper bounded.)

References

1. A. Day, *Characterizations of finite lattices that are bounded-homomorphic images or sublattices of free lattices*, Canad. J. Math **31** (1979), 69–78.
2. R. P. Dilworth, *The structure of relatively complemented lattices*, Ann. of Math. **51** (1950), 348–359.

3. R. Freese, *Finitely presented lattices: canonical forms and the covering relation*, Trans. Amer. Math. Soc. **312** (1989), 841–860.
4. R. Freese and J. B. Nation, *Covers in free lattices*, Trans. Amer. Math. Soc. **288** (1985), 1–42.
5. G. Grätzer and E. T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.
6. B. Jónsson and J. B. Nation, *A report on sublattices of a free lattice*, Coll. Math. Soc. János Bolyai **17** (1977), 233–257.
7. R. McKenzie, *Equational bases and non-modular lattice varieties*, Trans. Amer. Math. Soc. **174** (1972), 1–43.
8. J. B. Nation, *Lattice varieties covering* $V(L_1)$, Algebra Universalis **23** (1986), 132–166.
9. J. B. Nation, *An approach to lattice varieties of finite height*, Algebra Universalis **27** (1990), 521–543.

11. Geometric Lattices

Many's the time I've been mistaken
And many times confused . . . .
–Paul Simon

Now let us consider how we might use lattices to describe elementary geometry. There are two basic aspects of geometry: *incidence*, involving such statements as "the point p lies on the line l," and *measurement*, involving such concepts as angles and length. We will restrict our attention to incidence, which is most naturally stated in terms of lattices.

What properties should a *geometry* have? Without being too formal, surely we would want to include the following.

(1) The elements of a geometry (points, lines, planes, etc.) are subsets of a given set P of points.
(2) The set P of all points is an element of the geometry, and the intersection of any collection of elements is again one.
(3) There is a dimension function on the elements of the geometry, satisfying some sort of reasonable conditions.

If we order the elements of a geometry by set inclusion, then we obtain a lattice in which the atoms correspond to points of the geometry, every element is a join of atoms, and there is a well-behaved dimension function defined. With a little more care we can show that "well-behaved" means "semimodular" (recall Theorem 9.6). On the other hand, there is no harm if we allow some elements to have infinite dimension.

Accordingly, we define a *geometric lattice* to be an algebraic semimodular lattice in which every element is a join of atoms. As we have already described, the points, lines, planes, etc. (and the empty set) of a finite dimensional Euclidean geometry ($\Re^n$) form a geometric lattice. Other examples are the lattice of all subspaces of a vector space, and the lattice **Eq** X of equivalence relations on a set X. More examples are included in the exercises.[1]

We should note here that the geometric dimension of an element is generally one less than the lattice dimension δ: points are elements with $\delta(p) = 1$, lines are

[1] The basic properties of geometric lattices were developed by Garrett Birkhoff in the 1930's [2]. Similar ideas were pursued by K. Menger, F. Alt and O. Schreiber at about the same time [10]. Traditionally, geometric lattices were required to be finite dimensional, meaning $\delta(1) = n < \infty$. The last two examples show that this restriction is artificial.

elements with $\delta(l) = 2$, and so forth.

A lattice is said to be *atomistic* if every element is a join of atoms.

Theorem 11.1. *The following are equivalent.*

(1) *$\mathcal{L}$ is a geometric lattice.*
(2) *$\mathcal{L}$ is an upper continuous, atomistic, semimodular lattice.*
(3) *$\mathcal{L}$ is isomorphic to the lattice of ideals of an atomistic, semimodular, principally chain finite lattice.*

Proof. Every algebraic lattice is upper continuous, so (1) implies (2).

For (2) implies (3), we first note that the atoms of an upper continuous lattice are compact. For if $a \succ 0$ and $a \nleq \bigvee F$ for every finite $F \subseteq U$, then by Theorem 3.7 we have $a \wedge \bigvee U = \bigvee (a \wedge \bigvee F) = 0$, whence $a \nleq \bigvee U$. Thus in a lattice $\mathcal{L}$ satisfying condition (2), the compact elements are precisely the elements which are the join of finitely many atoms, in other words (using semimodularity) the finite dimensional elements. Let $\mathcal{K}$ denote the ideal of all finite dimensional elements of $\mathcal{L}$. Then $\mathcal{K}$ is a semimodular principally chain finite sublattice of $\mathcal{L}$, and it is not hard to see that the map $\phi : \mathcal{L} \to \mathcal{I}(\mathcal{K})$ by $\phi(x) = x/0 \, \cap \, \mathcal{K}$ is an isomorphism.

Finally, we need to show that if $\mathcal{K}$ is a semimodular principally chain finite lattice with every element the join of atoms, then $\mathcal{I}(\mathcal{K})$ is a geometric lattice. Clearly $\mathcal{I}(\mathcal{K})$ is algebraic, and every ideal is the join of the elements, and hence the atoms, it contains. It remains to show that $\mathcal{I}(\mathcal{K})$ is semimodular.

Suppose $I \succ I \cap J$ in $\mathcal{I}(\mathcal{K})$. Fix an atom $a \in I - J$. Then $I = (I \cap J) \vee a/0$, and hence $I \vee J = a/0 \vee J$. Let x be any element in $(I \vee J) - J$. Since $x \in I \vee J$, there exists $j \in J$ such that $x \leq a \vee j$. Because $\mathcal{K}$ is semimodular, $a \vee j \succ j$. On the other hand, every element of $\mathcal{K}$ is a join of finitely many atoms, so $x \notin J$ implies there exists an atom $b \leq x$ with $b \notin J$. Now $b \leq a \vee j$ and $b \nleq j$, so $b \vee j = a \vee j$, whence $a \leq b \vee j$. Thus $b/0 \vee J = I \vee J$; *a fortiori* it follows that $x/0 \vee J = I \vee J$. As this holds for every $x \in (I \vee J) - J$, we have $I \vee J \succ J$, as desired. $\square$

At the heart of the preceding proof is the following little argument: *if $\mathcal{L}$ is semimodular, a and b are atoms of $\mathcal{L}$, $t \in L$, and $b \leq a \vee t$ but $b \nleq t$, then $a \leq b \vee t$.* It is useful to interpret this property in terms of closure operators.

A closure operator Γ has the *exchange property* if $y \in \Gamma(B \cup \{x\})$ and $y \notin \Gamma(B)$ implies $x \in \Gamma(B \cup \{y\})$. Examples of algebraic closure operators with the exchange property include the span of a set of vectors in a vector space, the geometric closure of a set of points in Euclidean space, and the convex closure of a set of points in Euclidean space. More generally, we have the following representation theorem for geometric lattices, due to Saunders Mac Lane [9].

Theorem 11.2. *A lattice $\mathcal{L}$ is geometric if and only if $\mathcal{L}$ is isomorphic to the lattice of closed sets of an algebraic closure operator with the exchange property.*

Proof. Given a geometric lattice $\mathcal{L}$, we can define a closure operator Γ on the set A

of atoms of $\mathcal{L}$ by

$$\Gamma(X) = \{a \in A : a \leq \vee X\}.$$

Since the atoms are compact, this is an algebraic closure operator. By the little argument above, Γ has the exchange property. Because every element is a join of atoms, the map $\phi : \mathcal{L} \to \mathcal{C}_\Gamma$ given by $\phi(x) = \{a \in A : a \leq x\}$ is an isomorphism.

Now assume we have an algebraic closure operator Γ with the exchange property. Then $\mathcal{C}_\Gamma$ is an algebraic lattice. The exchange property insures that the closure of a singleton, $\Gamma(x)$, is an atom of $\mathcal{C}_\Gamma$: if $y \in \Gamma(x)$, then $x \in \Gamma(y)$, so $\Gamma(x) = \Gamma(y)$. Clearly, for every closed set we have $B = \bigvee_{b \in B} \Gamma(B)$. It remains to show that $\mathcal{C}_\Gamma$ is semimodular.

Let B and C be closed sets with $B \succ B \cap C$. Then $B = \Gamma(\{x\} \cup (B \cap C))$ for any $x \in B - (B \cap C)$. Suppose $C < D \leq B \vee C = \Gamma(B \cup C)$, and let y be any element in $D - C$. Fix any element $x \in B - (B \cap C)$. Then $y \in \Gamma(C \cup \{x\}) = B \vee C$, and $y \notin \Gamma(C) = C$. Hence $x \in \Gamma(C \cup \{y\})$, and $B \leq \Gamma(C \cup \{y\}) \leq D$. Thus $D = B \vee C$, and we conclude that $\mathcal{C}_\Gamma$ is semimodular. $\square$

Now we turn our attention to the structure of geometric lattices.

Theorem 11.3. *Every geometric lattice is relatively complemented.*

Proof. Let $a < x < b$ in a geometric lattice. By upper continuity and Zorn's Lemma, there exists an element y maximal with respect to the properties $a \leq y \leq b$ and $x \wedge y = a$. Suppose $x \vee y < b$. Then there is an atom p with $p \leq b$ and $p \not\leq x \vee y$. By the maximality of y we have $x \wedge (y \vee p) > a$; hence there is an atom q with $q \leq x \wedge (y \vee p)$ and $q \not\leq a$. Now $q \leq y \vee p$ but $q \not\leq y$, so by our usual argument $p \leq q \vee y \leq x \vee y$, a contradiction. Thus $x \vee y = b$, and y is a relative complement of x in b/a. $\square$

Let $\mathcal{L}$ be a geometric lattice, and let $\mathcal{K}$ be the ideal of compact elements of $\mathcal{L}$. By Theorem 10.10, $\mathcal{K}$ is a direct sum of simple lattices, and by Theorem 11.1, $\mathcal{L} \cong \mathcal{I}(\mathcal{K})$. So what we need now is a relation between the ideal lattice of a direct sum and the direct product of the corresponding ideal lattices.

Lemma 11.4. *For any collection of lattices $\mathcal{L}_i$ $(i \in I)$, we have $\mathcal{I}(\sum \mathcal{L}_i) \cong \prod \mathcal{I}(\mathcal{L}_i)$.*

Proof. If we identify $\mathcal{L}_i$ with the set of all vectors in $\sum \mathcal{L}_i$ which are zero except in the i-th place, then there is a natural map $\phi : \mathcal{I}(\sum \mathcal{L}_i) \to \prod \mathcal{I}(\mathcal{L}_i)$ given by $\phi(J) = \langle J_i \rangle_{i \in I}$, where $J_i = \{x \in L_i : x \in J\}$. It will be a relatively straightforward argument to show that this is an isomorphism. Clearly $J_i \in \mathcal{I}(\mathcal{L}_i)$, and the map ϕ is order preserving.

Assume $J, K \in \mathcal{I}(\sum \mathcal{L}_i)$ with $J \not\leq K$, and let $x \in J - K$. There exists an i_0 such that $x_{i_0} \notin K$, and hence $J_{i_0} \not\leq K_{i_0}$, whence $\phi(J) \not\leq \phi(K)$. Thus $\phi(J) \leq \phi(K)$ if and only if $J \leq K$, and ϕ is one-to-one.

It remains to show that ϕ is onto. Given $\langle T_i \rangle_{i \in I} \in \prod \mathcal{I}(\mathcal{L}_i)$, let $J = \{x \in \sum L_i : x_i \in T_i \text{ for all } i\}$. Then $J \in \mathcal{I}(\sum \mathcal{L}_i)$, and it is not hard to see that $J_i = T_i$ for all i, and hence $\phi(J) = \langle T_i \rangle_{i \in I}$, as desired. $\square$

So we are left with the task of describing the lattice of ideals of a simple semimodular principally chain finite lattice in which every element is a join of atoms. If $\mathcal{L} = \mathcal{I}(\mathcal{K})$ where $\mathcal{K}$ is such a lattice, then $\mathcal{L}$ is subdirectly irreducible: the unique minimal congruence μ is generated by collapsing all the finite dimensional elements to zero. So if $\mathcal{K}$ is finite dimensional (whence $\mathcal{L} \cong \mathcal{K}$), then $\mathcal{L}$ is simple, and it may be otherwise, as is the case with $\mathbf{Eq}\ X$. On the other hand, if $\mathcal{K}$ is modular and infinite dimensional, then μ will identify only those pairs (a, b) such that $a \vee b / a \wedge b$ is finite dimensional, and so $\mathcal{L}$ will not be simple. Summarizing, we have the following result.

Theorem 11.5. *Every geometric lattice is a direct product of subdirectly irreducible geometric lattices. Every finite dimensional geometric lattice is a direct product of simple geometric lattices.*

The finite dimensional case of Theorem 11.5 should be credited to Dilworth [3], and the extension is due to J. Hashimoto [6]. The best version of Hashimoto's theorem states that *a complete, weakly atomic, relatively complemented lattice is a direct product of subdirectly irreducible lattices.* A nice variation, due to L. Libkin [8], is that *every atomistic algebraic lattice is a direct product of directly indecomposable (atomistic algebraic) lattices.*

Before going on to modular geometric lattices, we should mention one of the most intriguing problems in combinatorial lattice theory. Let $\mathcal{L}$ be a finite geometric lattice, and let

$$w_k = |\{x \in L : \delta(x) = k\}|.$$

The *unimodal conjecture* states that there is always an integer m such that

$$1 = w_0 \leq w_1 \leq \dots w_{m-1} \leq w_m \geq w_{m+1} \geq \dots w_{n-1} \geq w_n = 1.$$

This is true if $\mathcal{L}$ is modular, and also for $\mathcal{L} = \mathbf{Eq}\ X$ with X finite ([5] and [7]). It is known that $w_1 \leq w_k$ always holds for for $1 \leq k < n$ ([1] and [4]). But a general resolution of the conjecture still seems to be a long way off.

Exercises for Chapter 11

1. Let $\mathcal{L}$ be a finite geometric lattice, and let F be a nonempty order filter on $\mathcal{L}$ (i.e., $x \geq f \in F$ implies $x \in F$). Show that the lattice $\mathcal{L}'$ obtained by identifying all the elements of F (a join semilattice congruence) is geometric.

2. Draw the following geometric lattices and their corresponding geometries:
   (a) $\mathbf{Eq}\ \mathbf{4}$,
   (b) $\mathbf{Sub}\ (Z_2)^3$, the lattice of subspaces of a 3-dimensional vector space over Z_2.

3. Show that each of the following is an algebraic closure operator on $\Re^n$ with the exchange property, and interpret them geometrically.
   (a) $\text{Span}(A) = \{\sum_{i=1}^{k} \lambda_i a_i : k \geq 1,\ a_i \in A \cup \{0\}\}$

(b) $\Gamma(A) = \{\sum_{i=1}^{k} \lambda_i a_i : k \geq 1,\ a_i \in A,\ \sum_{i=1}^{k} \lambda_i = 1\}$
(c) $\Delta(A) = \{\sum_{i=1}^{k} \lambda_i a_i : k \geq 1,\ a_i \in A,\ \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0\}$

4. Let G be a simple graph (no loops or multiple edges), and let X be the set of all edges of G. Define $S \subseteq X$ to be *closed* if whenever S contains all but one edge of a cycle, then it contains the entire cycle. Verify that the corresponding closure operator E is an algebraic closure operator with the exchange property. The lattice of E-closed subsets is called the *edge lattice* of G. Find the edge lattices of the graphs in Figure 11.1.

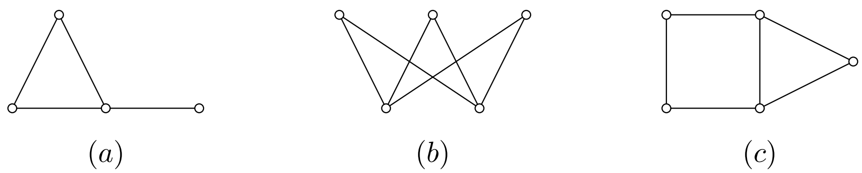


FIGURE 11.1

5. Show that the lattice for plane Euclidean geometry ($\Re^2$) is not modular. (Hint: Use two parallel lines and a point on one of them.)

6. (a) Let P and L be nonempty sets, which we will think of as "points" and "lines" respectively. Suppose we are given an arbitrary incidence relation $\in$ on $P \times L$. Then we can make $P \cup L \cup \{0, 1\}$ into a partially ordered set $\mathcal{K}$ in the obvious way, interpreting $p \in l$ as $p \leq l$. When is $\mathcal{K}$ a lattice? atomistic? semimodular? modular? subdirectly irreducible?

(b) Compare these results with Hilbert's axioms for a plane geometry.

(i) There exists at least one line.
(ii) On each line there exist at least two points.
(iii) Not all points are on the same line.
(iv) There is one and only one line passing through two given distinct points.

7. Let $\mathcal{L}$ be a geometric lattice, and let A denote the set of atoms of $\mathcal{L}$. A subset $S \subseteq A$ is *independent* if $p \not\leq \bigvee(S - \{p\})$ for all $p \in S$. A subset $B \subseteq A$ is a *basis* for $\mathcal{L}$ if B is independent and $\vee B = 1$.

(a) Prove that $\mathcal{L}$ has a basis.
(b) Prove that if B and C are bases for $\mathcal{L}$, then $|B| = |C|$.
(c) Show that the sublattice generated by an independent set S is isomorphic to the lattice of all finite subsets of S.

8. A lattice is *atomic* if for every $x > 0$ there exists $a \in L$ with $x \geq a \succ 0$. Prove that every element of a complete, relatively complemented, atomic lattice is a join of atoms.

9. Let I be an infinite set, and let $X = \{p_i : i \in I\} \dot{\cup} \{q_i : i \in I\}$. Define a subset S of X to be closed if $S = X$ or, for all i, at most one of p_i, q_i is in S. Let $\mathcal{L}$ be the lattice of all closed subsets of X.

(a) Prove that $\mathcal{L}$ is a relatively complemented algebraic lattice with every element the join of atoms.

(b) Show that the compact elements of $\mathcal{L}$ do not form an ideal.

(This example shows that the semimodularity hypothesis of Theorem 11.1 cannot be omitted.)

10. Prove that **Eq** X is relatively complemented and simple.

References

1. J. G. Basterfield and L. M. Kelly, *A characterization of sets of n points which determine n hyperplanes*, Proc. Camb. Phil. Soc. **64** (1968), 585–588.
2. G. Birkhoff, *Abstract linear dependence and lattices*, Amer. J. Math. **57** (1935), 800–804.
3. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. **51** (1950), 161–166.
4. C. Greene, *A rank inequality for finite geometric lattices*, J. Comb. Theory **9** (1970), 357–364.
5. L. H. Harper, *The morphology of partially ordered sets*, J. Combin. Theory Ser. A **17** (1974), 44–58.
6. J. Hashimoto, *Direct, subdirect decompositions and congruence relations*, Osaka J. Math. **9** (1957), 87–112.
7. W. N. Hsieh and D. J. Kleitman, *Normalized matching in direct products of partial orders*, Stud. Appl. Math. **52** (1973), 258–289.
8. L. Libkin, *Direct decompositions of atomistic algebraic lattices*, Algebra Universalis **33** (1995), 127–135.
9. S. Mac Lane, *A lattice formulation for transcendence degrees and p-bases*, Duke Math. J. **4** (1938), 455–468.
10. K. Menger, F. Alt, and O. Schreiber, *New foundations of projective and affine geometry*, Ann. of Math. **37** (1936), 456–482.

12. Complemented Modular Lattices

Lookout, the saints are comin' through
And it's all over now, Baby Blue.
–Bob Dylan

Traditionally, complemented modular lattices are the *alii*[1] of lattices, and subdirectly irreducible geometric modular lattices are the *alii nui*[2]. In this chapter we will see why.

Lemma 12.1. *Every complemented modular lattice is relatively complemented.*

Proof. Let c be a complement of x in a modular lattice, and let $a \geq x \geq b$. Consider the element $z = a \wedge (b \vee c)$, and note that $z = b \vee (a \wedge c)$ by modularity. Then $x \wedge z = x \wedge (b \vee c) = b \vee (x \wedge c) = b \vee 0 = b$, and dually $x \vee z = a$. Thus z is a relative complement of x in a/b. $\square$

Theorem 12.2. *Every algebraic complemented modular lattice is geometric.*

Proof. Let $\mathcal{L}$ be an algebraic complemented modular lattice. We need to show that the atoms of $\mathcal{L}$ are join dense, i.e., $a > b$ implies that there is an atom p with $p \leq a$ and $p \nleq b$. But we know that $\mathcal{L}$ is weakly atomic, so there exist elements c, d such that $a \geq c \succ d \geq b$. Let p be a relative complement of d in $c/0$. Then $c/d = (p \vee d)/d \cong p/(p \wedge d) = p/0$, whence p is an atom. Also $p \leq c \leq a$, while $p \nleq d$ implies $p \nleq b$. $\square$

The next result, though not needed in the sequel, is quite nice. It is due to Bjarni Jónsson [2], extending O. Frink [1].

Theorem 12.3. *Every complemented modular lattice can be embedded in a geometric modular lattice.*

Sketch of Proof. Let $\mathcal{L}$ be a complemented modular lattice. Let $\mathcal{F}(\mathcal{L})$ denote the lattice of filters of $\mathcal{L}$ ordered by reverse set inclusion, i.e., $F \leq G$ iff $F \supseteq G$. Since $\mathcal{L}$ has a 0, every filter of $\mathcal{L}$ is contained in a maximal (w.r.t. set inclusion) filter; hence $\mathcal{F}(\mathcal{L})$ is atomic (every nonzero element contains an atom). However, $\mathcal{F}(\mathcal{L})$ is dually algebraic, so we form the ideal lattice $\mathcal{I}(\mathcal{F}(\mathcal{L}))$, which is algebraic and is still atomic (but it may not be atomistic). Let $\mathfrak{A}$ denote the set of atoms of $\mathcal{I}(\mathcal{F}(\mathcal{L}))$, and let $A = \bigvee \mathfrak{A}$, so that A is the ideal of $\mathcal{F}(\mathcal{L})$ generated by the maximal filters. Then the

[1] Hawaiian chiefs.
[2] Hawaiian big chiefs.

sublattice $\mathcal{K} = A/0$ of $\mathcal{I}(\mathcal{F}(\mathcal{L}))$ is modular, algebraic, and atomistic, whence it is geometric.

It remains to show that $\mathcal{L}$ is embedded into $\mathcal{K}$ by the map $h(x) = \bigvee\{F \in \mathfrak{A} : x \in F\}$. This is a nontrivial (but doable) exercise. $\square$

Since every geometric modular lattice is a direct product of subdirectly irreducible ones, the following lemma comes in handy.

Lemma 12.4. *A geometric modular lattice is subdirectly irreducible if and only if for each pair of distinct atoms p, q there is a third atom r such that p, q and r generate a sublattice isomorphic to $\mathcal{M}_3$.*

Proof. First, suppose that the condition of the lemma holds in a geometric modular lattice $\mathcal{L}$. Since a geometric lattice is relatively complemented, every nontrivial congruence relation on $\mathcal{L}$ collapses some atom to 0. The condition implies that if one atom collapses to 0, then they all do. Thus the congruence μ generated by collapsing all the atoms to 0 is the unique minimum nontrivial congruence of $\mathcal{L}$, making it subdirectly irreducible. (In fact, μ collapses every finite dimensional quotient a/b of $\mathcal{L}$; unless the lattice is finite dimensional, this is not the universal congruence. See exercise 3.)

Conversely, assume that $\mathcal{L}$ is subdirectly irreducible. Then the compact elements of $\mathcal{L}$ form a simple, principally chain finite lattice; otherwise, as in the proof of Theorem 11.5, $\mathcal{L}^c$ would be a proper direct sum and $\mathcal{L}$ would be a direct product. Thus $|\mathcal{Q}_{\mathcal{L}^c}| = 1$, and for any two atoms p, q of $\mathcal{L}$ there is a sequence of atoms with

$$p = p_0 \mathrel{\underline{D}} p_1 \mathrel{\underline{D}} \dots \mathrel{\underline{D}} p_n = q.$$

Therefore, by induction, it will suffice to prove the following two claims.

(i) If p, q and r are atoms of $\mathcal{L}$ with $p \mathrel{\underline{D}} q \mathrel{\underline{D}} r$, then $p \mathrel{\underline{D}} r$.

(ii) If p and q are distinct atoms with $p \mathrel{\underline{D}} q$, then there is an atom s such that p, q and s generate a diamond.

Let us prove (ii) first. If p and q are distinct and $p \mathrel{\underline{D}} q$, then by definition there is an element $x \in L$ such that $p \leq q \vee x$ but $p \nleq q_* \vee x = x$. Note that, by modularity, $q \vee x \succ x$ and hence $p \vee x = q \vee x$. Set $s = x \wedge (p \vee q)$. Since $p \vee q$ has dimension 2 and $p \nleq x$, we have $s \succeq 0$. Now $p \nleq x$ implies $p \wedge s \leq p \wedge x = 0$. Similarly $q \nleq x$ and that implies $q \wedge s = 0$, while $p \neq q$ gives $p \wedge q = 0$. On the other hand, $p \vee s = (p \vee x) \wedge (p \vee q) = p \vee q$, and similarly $q \vee s = p \vee q$. Thus p, q and s generate an $\mathcal{M}_3$.

To prove (i), assume $p \mathrel{\underline{D}} q \mathrel{\underline{D}} r$. Without loss of generality, these are distinct, and hence by (ii) there exist atoms x and y such that p, q, x generate a diamond and q, r, y likewise. Set $z = (p \vee r) \wedge (x \vee y)$. Then

$$\begin{aligned} r \vee z &= (p \vee r) \wedge (r \vee x \vee y) \\ &= (p \vee r) \wedge (x \vee q \vee y) \\ &= (p \vee r) \wedge (p \vee q \vee y) \geq p. \end{aligned}$$

Now there are two possibilities. If $p \not\leq z$, then $p \underline{D} r$ *via* z, and we are done. So assume $p \leq z$. Then $p \leq x \vee y$ and $x \vee y$ has dimension 2, so $x \vee y = p \vee x = q \vee x$. But then $q \leq x \vee y$, whence $x \vee y = q \vee y$ also, i.e., the tops of the two diamonds coincide. In particular p, q and r join pairwise to $x \vee y$, so that again $p \underline{D} r$. □

It turns out that dimension plays an important role in subdirectly irreducible geometric lattices. We define the dimension of a geometric lattice $\mathcal{L}$ to be the length of a maximal chain in $\mathcal{L}$. Thus $\delta(\mathcal{L}) = \delta(1)$ if $\delta(1) = n < \infty$; more generally, $\delta(\mathcal{L}) = |B|$ where B is a basis for $\mathcal{L}$ (see exercise 11.7).

Of course **2** is the only geometric lattice with $\delta(\mathcal{L}) = 1$. Geometric lattices with $\delta(\mathcal{L}) = 2$ are isomorphic to $\mathcal{M}_\kappa$ for some cardinal κ, and these are simple whenever $\kappa > 2$.

Subdirectly irreducible geometric modular lattices with $\delta(\mathcal{L}) > 2$ correspond to projective geometries of geometric dimension ≥ 2. In particular, for $\delta(\mathcal{L}) = 3$ they correspond to projective planes. Projective planes come in two types: arguesian and nonarguesian. The nonarguesian projective planes are sort of strange: we can construct lots of examples of them, but there is no really good representation theorem for them.

On the other hand, a theorem of classical projective geometry translates as follows.

Theorem 12.5. *Every subdirectly irreducible geometric modular lattice with* $\delta(\mathcal{L}) \geq 4$ *is arguesian.*

Now we are ready for the best representation theorem of them all, due to Birkhoff and Frink (but based on older ideas from projective geometry). Recall that the lattice of subspaces of any vector space is a subdirectly irreducible geometric arguesian lattice.

Theorem 12.6. *Let* $\mathcal{L}$ *be a subdirectly irreducible geometric arguesian lattice with* $\delta(\mathcal{L}) = \kappa \geq 3$. *Then there is a division ring* D *such that* $\mathcal{L}$ *is isomorphic to the lattice of all subspaces of a* κ*-dimensional vector space over* D.

A later version of these notes will include a proof of Theorem 12.6, but not this one.

That's all, folks!

Exercises for Chapter 12

1. For what values of n is $\mathcal{M}_n \cong \mathbf{Sub}\, V$ for some vector space V?
2. The following steps carry you through the proof of Theorem 12.3.
   (a) Show that if $\mathcal{L}$ is a complete, upper continuous, modular lattice and 1 is a join of atoms, then $\mathcal{L}$ is geometric (i.e., algebraic and atomistic).
   (b) Find a finite semimodular lattice in which the atoms join to 1, but not every element is a join of atoms.

(c) For the map $h : \mathcal{L} \to \mathcal{K}$ given in the text, $h(x)$ is an ideal of $\mathcal{F}(\mathcal{L})$, and hence a set of filters. Show that $F \in h(x)$ iff F is the intersection of finitely many maximal (w.r.t. set inclusion) filters and $x \in F$. Note that this implies that h preserves order.

(d) Use (c) to show that $h(x \wedge y) = h(x) \wedge h(y)$.

(e) Show that, in order to prove that h preserves joins, it suffices to prove that $h(x \vee y) \leq h(x) \vee h(y)$ whenever $x \wedge y = 0$.

(f) Prove that if $x \wedge y = 0$ and F is a maximal filter with $F \in h(x \vee y)$, then there exist $G \in h(x)$ and $H \in h(y)$ such that $F \leq G \vee H$, i.e., $F \supseteq G \cap H$. Thus h preserves joins (by (e)).

3. Define a relation ξ on a modular lattice $\mathcal{L}$ by $\langle a, b \rangle \in \xi$ iff $(a \vee b)/(a \wedge b)$ is finite dimensional. Show that $\xi \in \mathbf{Con}\, \mathcal{L}$. Give examples to show that ξ can be 0, 1 or neither in $\mathbf{Con}\, \mathcal{L}$.

References

1. O. Frink, *Complemented modular lattices and projective spaces of infinite dimension*, Trans. Amer. Math. Soc. **60** (1946), 452–467.
2. B. Jónsson, *Modular lattices and Desargues' theorem*, Math. Scand. **2** (1954), 295–314.
3. J. von Neumann, *Continuous Geometries*, I. Halperin, ed., Princeton University Press, Princeton, 1960.

Appendix 1: Cardinals, Ordinals and Universal Algebra

In these notes we are assuming you have a working knowledge of cardinals and ordinals. Just in case, this appendix will give an informal summary of the most basic part of this theory. We also include an introduction to the terminology of universal algebra.

1. Ordinals

Let C be a well ordered set, i.e., a chain satisfying the descending chain condition (DCC). A *segment* of C is a proper ideal of C, which (because of the DCC) is necessarily of the form $\{c \in C : c < d\}$ for some $d \in C$.

Lemma. *Let C and D be well ordered sets. Then*

(1) *C is not isomorphic to any segment of itself.*
(2) *Either $C \cong D$, or C is isomorphic to a segment of D, or D is isomorphic to a segment of C.*

We say that two well ordered sets have the same *type* if $C \cong D$. An *ordinal* is an order type of well ordered sets. These are usually denoted by lower case Greek letters: α, β, γ, etc. For example, ω denotes the order type of the natural numbers, which is the smallest infinite ordinal. We can order ordinals by setting $\alpha \leq \beta$ if $\alpha \cong \beta$ or α is isomorphic to a segment of β. There are too many ordinals in the class of all ordinals to call this an ordered set without getting into set theoretic paradoxes, but we can say that locally it behaves like one big well ordered set.

Theorem. *Let β be an ordinal, and let B be the set of all ordinals α with $\alpha < \beta$, ordered by $\leq$. Then $B \cong \beta$.*

For example, ω is isomorphic to the collection of all finite ordinals.

Recall that the Zermelo well ordering principle (which is equivalent to the Axiom of Choice) says that every set can be well ordered. Another way of putting this is that every set can be indexed by ordinals,

$$X = \{x_\alpha : \alpha < \beta\}$$

for some β. Transfinite induction is a method of proof which involves indexing a set by ordinals, and then applying induction on the indices. This makes sense because the indices satisfy the DCC.

In doing transfinite induction, it is important to distinguish two types of ordinals. β is a *successor* ordinal if $\{\alpha : \alpha < \beta\}$ has a largest element. Otherwise, β is called a *limit* ordinal. For example, every finite ordinal is a successor ordinal, and ω is a limit ordinal.

2. Cardinals

We say that two sets X and Y have the same *cardinality*, written $|X| = |Y|$, if there exists a one-to-one onto map $f : X \to Y$. It is easy to see that "having the same cardinality" is an equivalence relation on the class of all sets, and the equivalence classes of this relation are called *cardinal numbers*. We will use lower case german letters such as $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{p}$ to denote unidentified cardinal numbers.

We order cardinal numbers as follows. Let X and Y be sets with $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$. Put $\mathfrak{m} \leq \mathfrak{n}$ if there exists a one-to-one map $f : X \rightarrowtail Y$ (equivalently, if there exists an onto map $g : Y \twoheadrightarrow X$). The Cantor-Bernstein theorem says that this relation is anti-symmetric: if $\mathfrak{m} \leq \mathfrak{n} \leq \mathfrak{m}$, then $\mathfrak{m} = \mathfrak{n}$, which is the hard part of showing that it is a partial order.

Theorem. *Let $\mathfrak{m}$ be any cardinal. Then there is a least ordinal α with $|\alpha| = \mathfrak{m}$.*

Theorem. *Any set of cardinal numbers is well ordered.*[1]

Now let $|X| = \mathfrak{m}$ and $|Y| = \mathfrak{n}$ with X and Y disjoint. We introduce operations on cardinals (which agree with the standard operations in the finite case) as follows.

$$\begin{aligned} \mathfrak{m} + \mathfrak{n} &= |X \cup Y| \\ \mathfrak{m} \cdot \mathfrak{n} &= |X \times Y| \\ \mathfrak{m}^{\mathfrak{n}} &= |\{f : Y \to X\}| \end{aligned}$$

It should be clear how to extend $+$ and $\cdot$ to arbitrary sums and products.

The basic arithmetic of infinite cardinals is fairly simple.

Theorem. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals. Then*

(1) $\mathfrak{m} + \mathfrak{n} = \mathfrak{m} \cdot \mathfrak{n} = \max\{\mathfrak{m}, \mathfrak{n}\}$,
(2) $2^{\mathfrak{m}} > \mathfrak{m}$.

The finer points of the arithmetic can get complicated, but that will not bother us here. The following facts are used frequently.

Theorem. *Let X be an infinite set, $\mathcal{P}(X)$ the lattice of subsets of X, and $\mathcal{P}_f(X)$ the lattice of finite subsets of X. Then $|\mathcal{P}(X)| = 2^{|X|}$ and $|\mathcal{P}_f(X)| = |X|$.*

A fine little book [2] by Irving Kaplansky, *Set Theory and Metric Spaces*, is easy reading and contains the proofs of the theorems above. The book *Introduction to Modern Set Theory* by Judith Roitman [4] is recommended for a slightly more advanced introduction.

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[1] Again, there are too many cardinals to talk about the "set of all cardinals".

3. Universal Algebra

Once you have seen enough different kinds of algebras: vector spaces, groups, rings, semigroups, lattices, even semilattices, you should be driven to abstraction. The proper abstraction in this case is the general notion of an "algebra". *Universal algebra* is the study of the properties which different types of algebras have in common. Historically, lattice theory and universal algebra developed together, more like Siamese twins than cousins. In these notes we do not assume you know much universal algebra, but where appropriate we do use its terminology.

An *operation* on a set A is just a function $f : A^n \to A$ for some $n \in \omega$. An *algebra* is a system $\mathcal{A} = \langle A; \mathcal{F} \rangle$ where A is a nonempty set and $\mathcal{F}$ is a set of operations on A. Note that we allow infinitely many operations, but each has only finitely many arguments. For example, lattices have two binary operations, $\wedge$ and $\vee$. We use different fonts to distinguish between an algebra and the set of its elements, e.g., $\mathcal{A}$ and A.

Many algebras have distinguished elements, or constants. For example, groups have a unit element e, rings have both 0 and 1. Technically, these constants are nullary operations (with no arguments), and are included in the set $\mathcal{F}$ of operations. However, in these notes we sometimes revert to a more old-fashioned notation and write them separately, as $\mathcal{A} = \langle A; \mathcal{F}, \mathcal{C} \rangle$, where $\mathcal{F}$ is the set of operations with at least one argument and $\mathcal{C}$ is the set of constants. There is no requirement that constants with different names, e.g., 0 and 1, be distinct.

A *subalgebra* of $\mathcal{A}$ is a subset S of A which is closed under the operations, i.e., if $s_1, \ldots, s_n \in S$ and $f \in \mathcal{F}$, then $f(s_1, \ldots, s_n) \in S$. This means in particular that all the constants of $\mathcal{A}$ are contained in S. If $\mathcal{A}$ has no constants, then we allow the empty set as a subalgebra (even though it is not properly an algebra). Thus the empty set is a sublattice of a lattice, but not a subgroup of a group. A nonempty subalgebra S of $\mathcal{A}$ can of course be regarded as an algebra $\mathcal{S}$ of the same type as $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are algebras with the same operation symbols (including constants), then a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $h : A \to B$ which preserves the operations, i.e., $h(f(a_1, \ldots, a_n)) = f(h(a_1), \ldots, h(a_n))$ for all $a_1, \ldots, a_n \in A$ and $f \in \mathcal{F}$. This includes that $h(c) = c$ for all $c \in \mathcal{C}$.

A homomorphism which is one-to-one is called an *embedding*, and sometimes written $h : \mathcal{A} \rightarrowtail \mathcal{B}$ or $h : \mathcal{A} \leq \mathcal{B}$. A homomorphism which is both one-to-one and onto is called an *isomorphism*, denoted $h : \mathcal{A} \cong \mathcal{B}$.

These notions directly generalize notions which should be perfectly familiar to you for say groups or rings. Note that we have given only terminology, but no results. The basic theorems of universal algebra are included in the text, either in full generality, or for lattices in a form which is easy to generalize. For deeper results in universal algebra, there are several nice textbooks available, including *A Course in Universal Algebra* by S. Burris and H. P. Sankappanavar [1], and *Algebras, Lattices, Varieties* by R. McKenzie, G. McNulty and W. Taylor [3].

References

1. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1980.
2. I. Kaplansky, *Set Theory and Metric Spaces*, Allyn and Bacon, Boston, 1972.
3. R. McKenzie, G. McNulty and W. Taylor, *Algebras, Lattices, Varieties*, vol. I, Wadsworth and Brooks-Cole, Belmont, CA, 1987.
4. J. Roitman, *Introduction to Modern Set Theory*, Wiley, New York, 1990.

Appendix 2: The Axiom of Choice

In this appendix we want to prove Theorem 1.5.

Theorem 1.5. *The following set theoretic axioms are equivalent.*

(1) (Axiom of Choice) *If X is a nonempty set, then there is a map $\phi : \mathfrak{P}(X) \to X$ such that $\phi(A) \in A$ for every nonempty $A \subseteq X$.*
(2) (Zermelo well-ordering principle) *Every nonempty set admits a well-ordering (a total order satisfying the DCC).*
(3) (Hausdorff maximality principle) *Every chain in an ordered set $\mathcal{P}$ can be embedded in a maximal chain.*
(4) (Zorn's Lemma) *If every chain in an ordered set $\mathcal{P}$ has an upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal element.*
(5) *If every chain in an ordered set $\mathcal{P}$ has a least upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal element.*

Let us start by proving the equivalence of (1), (2) and (4).

(4) $\implies$ (2): Given a nonempty set X, let $\mathcal{Q}$ be the collection of all pairs (Y, R) such that $Y \subseteq X$ and R is a well ordering of Y, i.e., $R \subseteq Y \times Y$ is a total order satisfying the DCC. Order $\mathcal{Q}$ by $(Y, R) \sqsubseteq (Z, S)$ if $Y \subseteq Z$ and $R \subseteq S$. In order to apply Zorn's Lemma, check that if $\{(Y_\alpha, R_\alpha) : \alpha \in A\}$ is a chain in $\mathcal{Q}$, then $(\overline{Y}, \overline{R}) = (\bigcup Y_\alpha, \bigcup R_\alpha) \in \mathcal{Q}$ and $(Y_\alpha, R_\alpha) \sqsubseteq (\overline{Y}, \overline{R})$ for every $\alpha \in A$, and so $(\overline{Y}, \overline{R})$ is an upper bound for $\{(Y_\alpha, R_\alpha) : \alpha \in A\}$. Thus $\mathcal{Q}$ contains a maximal element (U, T). Moreover, we must have $U = X$. For otherwise we could choose an element $z \in X - U$, and then the pair (U', T') with $U' = U \cup \{z\}$ and $T' = T \cup \{(u, z) : u \in U\}$ would satisfy $(U, T) \sqsubset (U', T')$, a contradiction. Therefore T is a well ordering of $U = X$, as desired.

(2) $\implies$ (1): Given a well ordering $\leq$ of X, we can define a choice function ϕ on the nonempty subsets of X by letting $\phi(A)$ be the least element of A under the ordering $\leq$.

(1) $\implies$ (4): For a subset S of an ordered set $\mathcal{P}$, let S^u denote the set of all upper bounds of S, i.e., $S^u = \{x \in P : x \geq s \text{ for all } s \in S\}$.

Let $\mathcal{P}$ be an ordered set in which every chain has an upper bound. By the Axiom of Choice there is a function ϕ on the subsets of P such that $\phi(S) \in S$ for every nonempty $S \subseteq P$. We use the choice function ϕ to construct a function which assigns a strict upper bound to every subset of P which has one as follows: if $S \subseteq P$ and $S^u - S = \{x \in P : x > s \text{ for all } s \in S\}$ is nonempty, define $\gamma(S) = \phi(S^u - S)$.

Fix an element $x_0 \in P$. Let $\mathfrak{B}$ be the collection of all subsets $B \subseteq P$ satisfying the following properties.

(1) B is a chain.
(2) $x_0 \in B$.
(3) $x_0 \leq y$ for all $y \in B$.
(4) If A is a nonempty order ideal of B and $z \in B \cap (A^u - A)$, then $\gamma(A) \in B \cap z/0$.

The last condition says that if A is a proper ideal of B, then $\gamma(A)$ is in B, and moreover it is the least element of B strictly above every member of A.

Note that $\mathfrak{B}$ is nonempty, since $\{x_0\} \in \mathfrak{B}$.

Next, we claim that *if B and C are both in $\mathfrak{B}$, then either B is an order ideal of C or C is an order ideal of B.* Suppose not, and let $A = \{t \in B \cap C : t/0 \cap B = t/0 \cap C\}$. Thus A is the largest common ideal of B and C; it contains x_0, and by assumption is a proper ideal of both B and C. Let $b \in B - A$ and $c \in C - A$. Now B is a chain and A is an ideal of B, so $b \notin A$ implies $b > a$ for all $a \in A$, whence $b \in B \cap (A^u - A)$. Likewise $c \in C \cap (A^u - A)$. Hence by (4), $\gamma(A) \in B \cap C$. Moreover, since b was arbitrary in $B - A$, again by (4) we have $\gamma(A) \leq b$ for all $b \in B - A$, and similarly $\gamma(A) \leq c$ for all $c \in C - A$. Therefore

$$\gamma(A)/0 \cap B = A \cup \{\gamma(A)\} = \gamma(A)/0 \cap C$$

whence $\gamma(A) \in A$, contrary to the definition of γ.

It follows, that if B and C are in $\mathfrak{B}$, $b \in B$ and $c \in C$, and $b \leq c$, then $b \in C$.

Also, you can easily check that if $B \in \mathfrak{B}$ and $B^u - B$ is nonempty, then $B \cup \{\gamma(B)\} \in \mathfrak{B}$.

Now let $U = \bigcup_{B \in \mathfrak{B}} B$. We claim that $U \in \mathfrak{B}$. It is a chain because for any two elements b, $c \in U$ there exist B, $C \in \mathfrak{B}$ with $b \in B$ and $c \in C$; one of B and C is an ideal of the other, so both are contained in the larger set and hence comparable. Conditions (2) and (3) are immediate. If a nonempty ideal A of U has a strict upper bound $z \in U$, then $z \in C$ for some $C \in \mathfrak{B}$. By the observation above, A is an ideal of C, and hence the conclusion of (4) holds.

Now U is a chain in $\mathcal{P}$, and hence by hypothesis U has an upper bound x. On the other hand, $U^u - U$ must be empty, for otherwise $U \cup \{\gamma(U)\} \in \mathfrak{B}$, whence $\gamma(U) \in U$, a contradiction. Therefore $x \in U$ and x is maximal in $\mathcal{P}$. In particular, $\mathcal{P}$ has a maximal element, as desired.

Now we prove the equivalence of (3), (4) and (5).

(4) $\implies$ (5): This is obvious, since the hypothesis of (5) is stronger.

(5) $\implies$ (3): Given an ordered set $\mathcal{P}$, let $\mathcal{Q}$ be the set of all chains in $\mathcal{P}$, ordered by set containment. If $\{C_\alpha : \alpha \in A\}$ is a chain in $\mathcal{Q}$, then $\bigcup C_\alpha$ is a chain in $\mathcal{P}$ which is the least upper bound of $\{C_\alpha : \alpha \in A\}$. Thus $\mathcal{Q}$ satisfies the hypothesis of (5), and hence it contains a maximal element C, which is a maximal chain in $\mathcal{P}$.

(3) $\implies$ (4): Let $\mathcal{P}$ be an ordered set such that every chain in $\mathcal{P}$ has an upper bound in P. By (3), there is a maximal chain C in $\mathcal{P}$. If b is an upper bound for C, then in fact $b \in C$ (by maximality), and b is a maximal element of $\mathcal{P}$.

(There are many variations of the proof of Theorem 1.5, but it can always be arranged so that there is only one hard step, and the rest easy. The above version seems fairly natural.)

Appendix 3: Formal Concept Analysis

Exercise 13 of Chapter 2 is to show that a binary relation $R \subseteq A \times B$ induces a pair of closure operators, described as follows. For $X \subseteq A$, let

$$\sigma(X) = \{b \in B : x \mathrel{R} b \text{ for all } x \in X\}.$$

Similarly, for $Y \subseteq B$, let

$$\pi(Y) = \{a \in A : a \mathrel{R} y \text{ for all } y \in Y\}.$$

Then the composition $\pi\sigma : \mathfrak{P}(A) \to \mathfrak{P}(A)$ is a closure operator on A, given by

$$\pi\sigma(X) = \{a \in A : a \mathrel{R} b \text{ whenever } x \mathrel{R} b \text{ for all } x \in X\}.$$

Likewise, $\sigma\pi$ is a closure operator on B, and for $Y \subseteq B$,

$$\sigma\pi(Y) = \{b \in B : a \mathrel{R} b \text{ whenever } a \mathrel{R} y \text{ for all } y \in Y\}.$$

In this situation, the lattice of closed sets $\mathcal{C}_{\pi\sigma} \subseteq \mathfrak{P}(A)$ is dually isomorphic to $\mathcal{C}_{\sigma\pi} \subseteq \mathfrak{P}(B)$, and we say that R establishes a *Galois connection* between the $\pi\sigma$-closed subsets of A and the $\sigma\pi$-closed subsets of B.

Of course, $\mathcal{C}_{\pi\sigma}$ is a complete lattice. Moreover, every complete lattice can be represented *via* a Galois connection.

Theorem. *Let $\mathcal{L}$ be a complete lattice, A a join dense subset of L and B a meet dense subset of L. Define $R \subseteq A \times B$ by $a \mathrel{R} b$ if and only if $a \leq b$. Then, with σ and π defined as above, $\mathcal{L} \cong \mathcal{C}_{\pi\sigma}$ (and $\mathcal{L}$ is dually isomorphic to $\mathcal{C}_{\sigma\pi}$).*

In particular, for an arbitrary complete lattice, we can always take $A = B = L$. If $\mathcal{L}$ is algebraic, a more natural choice is $A = L^c$ and $B = M^*(\mathcal{L})$ (compact elements and completely meet irreducibles). If $\mathcal{L}$ is finite, the most natural choice is $A = J(\mathcal{L})$ and $B = M(\mathcal{L})$. Again the proof of this theorem is elementary.

Formal Concept Analysis is a method developed by Rudolf Wille and his colleagues in Darmstadt (Germany), whereby the philosophical Galois connection between objects and their properties is used to provide a systematic analysis of certain very general situations. Abstractly, it goes like this. Let G be a set of "objects" (*Gegenstände*) and M a set of relevant "attributes" (*Merkmale*). The relation $I \subseteq G \times M$ consists of all those pairs $\langle g, m\rangle$ such that g has the property m. A *concept* is a pair $\langle X, Y\rangle$ with $X \subseteq G$, $Y \subseteq M$, $X = \pi(Y)$ and $Y = \sigma(X)$. Thus

$\langle X, Y \rangle$ is a concept if X is the set of all elements with the properties of Y, and Y is exactly the set of properties shared by the elements of X. It follows (as in exercise 12, Chapter 2) that $X \in \mathcal{C}_{\pi\sigma}$ and $Y \in \mathcal{C}_{\sigma\pi}$. Thus if we order concepts by $\langle X, Y \rangle \leq \langle U, V \rangle$ iff $X \subseteq U$ (which is equivalent to $Y \supseteq V$), then we obtain a lattice $\mathfrak{B}(G, M, I)$ isomorphic to $\mathcal{C}_{\pi\sigma}$.

A small example will illustrate how this works. The rows of Table A1 correspond to seven fine musicians, and the columns to eight possible attributes (chosen by a musically trained sociologist). An $\times$ in the table indicates that the musician has that attribute.[1] The corresponding concept lattice is given in Figure A2, where the musicians are abbreviated by lower case letters and their attributes by capitals.

	Instrument	Classical	Jazz	Country	Black	White	Male	Female
J. S. Bach	×	×				×	×	
Rachmaninoff	×	×				×	×	
King Oliver	×		×		×		×	
W. Marsalis	×	×	×		×		×	
B. Holiday			×		×			×
Emmylou H.				×		×		×
Chet Atkins	×		×	×		×	×	

Table A1.

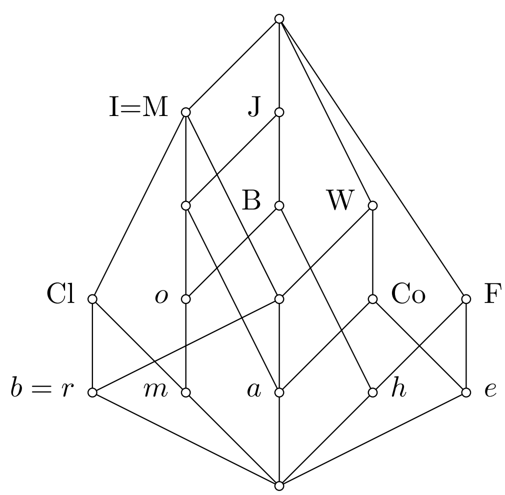

FIGURE A2

[1] To avoid confusion, androgynous rock stars were not included.

Formal concept analysis has been applied to hundreds of real situations outside of mathematics (e.g., law, medicine, psychology), and has proved to be a useful tool for understanding the relation between the concepts involved. Typically, these applications involve large numbers of objects and attributes, and computer programs have been developed to navigate through the concept lattice. A good brief introduction to concept analysis may be found in Wille [2] or [3], and the whole business is explained thoroughly in Ganter and Wille [1].

Likewise, the representation of a finite lattice as the concept lattice induced by the order relation between join and meet irreducible elements (i.e., $\leq$ restricted to $J(\mathcal{L}) \times M(\mathcal{L})$) provides and effective and tractable encoding of its structure. As an example of the method, let us show how one can extract the ordered set $\mathcal{Q}_{\mathcal{L}}$ such that **Con** $\mathcal{L} \cong \mathcal{O}(\mathcal{Q}(\mathcal{L}))$ from the table.

Given a finite lattice $\mathcal{L}$, for $g \in J(\mathcal{L})$ and $m \in M(\mathcal{L})$, define

$$\begin{aligned} g \nearrow m &\quad \text{if } g \not\leq m \text{ but } g \leq m^*, \text{ i.e., } g \leq n \text{ for all } n > m, \\ m \searrow g &\quad \text{if } m \not\geq g \text{ but } m \geq g_*, \text{ i.e., } m \geq h \text{ for all } h < g, \\ g \updownarrow m &\quad \text{if } g \nearrow m \text{ and } m \searrow g. \end{aligned}$$

Note that these relations can easily be added to the table of $J(\mathcal{L}) \times M(\mathcal{L})$.

These relations connect with the relation $\underline{D}$ of Chapter 10 as follows.

Lemma. *Let $\mathcal{L}$ be a finite lattice and $g, h \in J(\mathcal{L})$. Then $g \ \underline{D}\ h$ if and only if there exists $m \in M(\mathcal{L})$ such that $g \nearrow m \searrow h$.*

Proof. If $g \ \underline{D}\ h$, then there exists $x \in L$ such that $g \leq h \vee x$ but $g \not\leq h_* \vee x$. Let m be maximal such that $m \geq h_* \vee x$ but $m \not\geq g$. Then $m \in M(\mathcal{L})$, $g \leq m^*$, $m \geq h_*$ but $m \not\geq h$. Thus $g \nearrow m \searrow h$.

Conversely, suppose $g \nearrow m \searrow h$. Then $g \leq m^* \leq h \vee m$ while $g \not\leq m = h_* \vee m$. Therefore $g \ \underline{D}\ h$. □

As an example, the table for the lattice in Figure A2 is given in Table A3. This is a reduction of the original Table A1: $J(\mathcal{L})$ is a subset of the original set of objects, and likewise $M(\mathcal{L})$ is contained in the original attributes. Arrows indicating the relations $\nearrow$, $\searrow$ and $\updownarrow$ have been added. The Lemma allows us to calculate $\underline{D}$ quickly, and we find that $|\mathcal{Q}_{\mathcal{L}}| = 1$, whence $\mathcal{L}$ is simple.

	I=M	Cl	J	Co	B	W	F
b=r	$\times$	$\times$	$\updownarrow$	$\updownarrow$	$\searrow$	$\times$	$\updownarrow$
o	$\times$	$\updownarrow$	$\times$		$\times$	$\nearrow$	$\nearrow$
m	$\times$	$\times$	$\times$	$\searrow$	$\times$	$\updownarrow$	$\updownarrow$
h	$\updownarrow$	$\searrow$	$\times$	$\searrow$	$\times$	$\updownarrow$	$\times$
e	$\updownarrow$	$\searrow$	$\updownarrow$	$\times$	$\updownarrow$	$\times$	$\times$
a	$\times$	$\updownarrow$	$\times$	$\times$	$\updownarrow$	$\times$	$\updownarrow$

Table A3.

References

1. B. Ganter and R. Wille, *Formale Begriffsanalyse: Mathematische Grundlagen*, Springer-Verlag, Berlin-Heidelberg, 1996.
2. R. Wille, *Restructuring lattice theory: an approach based on hierarchies of concepts*, Ordered Sets, I. Rival, ed., Reidel, Dordrecht-Boston, 1982, pp. 445–470.
3. R. Wille, *Concept lattices and conceptual knowledge systems*, Computers and Mathematics with Applications **23** (1992), 493–515.