11. Geometric Lattices

Many's the time I've been mistaken
And many times confused ....
–Paul Simon

Now let us consider how we might use lattices to describe elementary geometry. There are two basic aspects of geometry: *incidence*, involving such statements as "the point p lies on the line l," and *measurement*, involving such concepts as angles and length. We will restrict our attention to incidence, which is most naturally stated in terms of lattices.

What properties should a *geometry* have? Without being too formal, surely we would want to include the following.

(1) The elements of a geometry (points, lines, planes, etc.) are subsets of a given set P of points.
(2) Each point $p \in P$ is an element of the geometry.
(3) The set P of all points is an element of the geometry, and the intersection of any collection of elements is again one.
(4) There is a dimension function on the elements of the geometry, satisfying some sort of reasonable conditions.

If we order the elements of a geometry by set inclusion, then we obtain a lattice in which the atoms correspond to points of the geometry, every element is a join of atoms, and there is a well-behaved dimension function defined. With a little more care we can show that "well-behaved" means "semimodular" (recall Theorem 9.6). On the other hand, there is no harm if we allow some elements to have infinite dimension.

Accordingly, we define a *geometric lattice* to be an algebraic semimodular lattice in which every element is a join of atoms. As we have already described, the points, lines, planes, etc. (and the empty set) of a finite dimensional Euclidean geometry ($\Re^n$) form a geometric lattice. Other examples are the lattice of all subspaces of a vector space, and the lattice **Eq** X of equivalence relations on a set X. More examples are included in the exercises.[1]

[1] The basic properties of geometric lattices were developed by Garrett Birkhoff in the 1930's [3]. Similar ideas were pursued by K. Menger, F. Alt and O. Schreiber at about the same time [12]. Traditionally, geometric lattices were required to be finite dimensional, meaning $\delta(1) = n < \infty$. The last two examples show that this restriction is artificial.

We should note here that the geometric dimension of an element is generally one less than the lattice dimension δ: points are elements with $\delta(p) = 1$, lines are elements with $\delta(l) = 2$, and so forth.

A lattice is said to be *atomistic* if every element is a join of atoms.

Theorem 11.1. *The following are equivalent.*

(1) *$\mathcal{L}$ is a geometric lattice.*
(2) *$\mathcal{L}$ is an upper continuous, atomistic, semimodular lattice.*
(3) *$\mathcal{L}$ is isomorphic to the lattice of ideals of an atomistic, semimodular, principally chain finite lattice.*

In fact, we will show that if $\mathcal{L}$ is a geometric lattice and $\mathcal{K}$ its set of finite dimensional elements, then $\mathcal{L} \cong \mathcal{I}(\mathcal{K})$ and $\mathcal{K}$ is the set of compact elements of $\mathcal{L}$.

Proof. Every algebraic lattice is upper continuous, so (1) implies (2).

For (2) implies (3), we first note that the atoms of an upper continuous lattice are compact. For if $a \succ 0$ and $a \nleq \bigvee F$ for every finite $F \subseteq U$, then by Theorem 3.7 we have $a \wedge \bigvee U = \bigvee (a \wedge \bigvee F) = 0$, whence $a \nleq \bigvee U$. Thus in a lattice $\mathcal{L}$ satisfying condition (2), the compact elements are precisely the elements that are the join of finitely many atoms, in other words (using semimodularity) the finite dimensional elements. Let $\mathcal{K}$ denote the ideal of all finite dimensional elements of $\mathcal{L}$. Then $\mathcal{K}$ is a semimodular principally chain finite sublattice of $\mathcal{L}$, and it is not hard to see that the map $\phi : \mathcal{L} \to \mathcal{I}(\mathcal{K})$ by $\phi(x) = \downarrow x \, \cap \, \mathcal{K}$ is an isomorphism.

Finally, we need to show that if $\mathcal{K}$ is a semimodular principally chain finite lattice with every element the join of atoms, then $\mathcal{I}(\mathcal{K})$ is a geometric lattice. Clearly $\mathcal{I}(\mathcal{K})$ is algebraic, and every ideal is the join of the elements, and hence the atoms, it contains. It remains to show that $\mathcal{I}(\mathcal{K})$ is semimodular.

Suppose $I \succ I \cap J$ in $\mathcal{I}(\mathcal{K})$. Fix an atom $a \in I - J$. Then $I = (I \cap J) \vee \downarrow a$, and hence $I \vee J = \downarrow a \vee J$. Let x be any element in $(I \vee J) - J$. Since $x \in I \vee J$, there exists $j \in J$ such that $x \leq a \vee j$. Because $\mathcal{K}$ is semimodular, $a \vee j \succ j$. On the other hand, every element of $\mathcal{K}$ is a join of finitely many atoms, so $x \notin J$ implies there exists an atom $b \leq x$ with $b \notin J$. Now $b \leq a \vee j$ and $b \nleq j$, so $b \vee j = a \vee j$, whence $a \leq b \vee j$. Thus $\downarrow b \vee J = I \vee J$; *a fortiori* it follows that $\downarrow x \vee J = I \vee J$. As this holds for every $x \in (I \vee J) - J$, we have $I \vee J \succ J$, as desired. $\square$

At the heart of the preceding proof is the following little argument: *if $\mathcal{L}$ is semimodular, a and b are atoms of $\mathcal{L}$, $t \in L$, and $b \leq a \vee t$ but $b \nleq t$, then $a \leq b \vee t$.* It is useful to interpret this property in terms of closure operators.

A closure operator Γ has the *exchange property* if $y \in \Gamma(B \cup \{x\})$ and $y \notin \Gamma(B)$ implies $x \in \Gamma(B \cup \{y\})$. Examples of algebraic closure operators with the exchange property include the span of a set of vectors in a vector space, and the geometric closure of a set of points in Euclidean space. More generally, we have the following representation theorem for geometric lattices, due to Saunders Mac Lane [11].

Theorem 11.2. *A lattice $\mathcal{L}$ is geometric if and only if $\mathcal{L}$ is isomorphic to the lattice of closed sets of an algebraic closure operator with the exchange property.*

Proof. Given a geometric lattice $\mathcal{L}$, we can define a closure operator Γ on the set A of atoms of $\mathcal{L}$ by

$$\Gamma(X) = \{a \in A : a \leq \bigvee X\}.$$

Since the atoms are compact, this is an algebraic closure operator. By the little argument above, Γ has the exchange property. Because every element is a join of atoms, the map $\phi : \mathcal{L} \to \mathcal{C}_\Gamma$ given by $\phi(x) = \{a \in A : a \leq x\}$ is an isomorphism.

Now assume we have an algebraic closure operator Γ with the exchange property. Then $\mathcal{C}_\Gamma$ is an algebraic lattice. The exchange property insures that the closure of a singleton, $\Gamma(x)$, is either the least element $\Gamma(\emptyset)$ or an atom of $\mathcal{C}_\Gamma$: if $y \in \Gamma(x)$, then $x \in \Gamma(y)$, so $\Gamma(x) = \Gamma(y)$. Clearly, for every closed set we have $B = \bigvee_{b \in B} \Gamma(B)$. It remains to show that $\mathcal{C}_\Gamma$ is semimodular.

Let B and C be closed sets with $B \succ B \cap C$. Then $B = \Gamma(\{x\} \cup (B \cap C))$ for any $x \in B - (B \cap C)$. Suppose $C < D \leq B \vee C = \Gamma(B \cup C)$, and let y be any element in $D - C$. Fix any element $x \in B - (B \cap C)$. Then $y \in \Gamma(C \cup \{x\}) = B \vee C$, and $y \notin \Gamma(C) = C$. Hence $x \in \Gamma(C \cup \{y\})$, and $B \leq \Gamma(C \cup \{y\}) \leq D$. Thus $D = B \vee C$, and we conclude that $\mathcal{C}_\Gamma$ is semimodular. □

Now we turn our attention to the structure of geometric lattices.

Theorem 11.3. *Every geometric lattice is relatively complemented.*

Proof. Let $a < x < b$ in a geometric lattice. By upper continuity and Zorn's Lemma, there exists an element y maximal with respect to the properties $a \leq y \leq b$ and $x \wedge y = a$. Suppose $x \vee y < b$. Then there is an atom p with $p \leq b$ and $p \not\leq x \vee y$. Note that $y \vee p \succ y$, wherefore $(x \vee y) \wedge (y \vee p) = y$. But then

$$x \wedge (y \vee p) = x \wedge (x \vee y) \wedge (y \vee p) x \wedge y = a\ ,$$

a contradiction. Thus $x \vee y = b$, and y is a relative complement of x in $[a, b]$.
□

Let $\mathcal{L}$ be a geometric lattice, and let $\mathcal{K}$ be the ideal of compact elements of $\mathcal{L}$. By Theorem 10.10, $\mathbf{K}$ is a direct sum of simple lattices, and by Theorem 11.1, $\mathcal{L} \cong \mathcal{I}(\mathbf{K})$. So what we need now is a relation between the ideal lattice of a direct sum and the direct product of the corresponding ideal lattices.

Lemma 11.4. *For any collection of lattices $\mathbf{K}_i$ $(i \in I)$, we have $\mathcal{I}(\sum \mathbf{K}_i) \cong \prod \mathcal{I}(\mathbf{K}_i)$.*

Proof. If we identify $\mathbf{K}_i$ with the set of all vectors in $\sum \mathbf{K}_i$ that are zero except in the i-th place, then there is a natural map $\phi : \mathcal{I}(\sum \mathbf{K}_i) \to \prod \mathcal{I}(\mathbf{K}_i)$ given by $\phi(J) = \langle J_i \rangle_{i \in I}$, where $J_i = \{x \in L_i : x \in J\}$. It will be a relatively straightforward argument to show that this is an isomorphism. Clearly $J_i \in \mathcal{I}(\mathbf{K}_i)$, and the map ϕ is order preserving.

Assume $J, H \in \mathcal{I}(\sum \mathbf{K}_i)$ with $J \nleq H$, and let $x \in J - H$. There exists an i_0 such that $x_{i_0} \notin H$, and hence $J_{i_0} \nleq H_{i_0}$, whence $\phi(J) \nleq \phi(H)$. Thus $\phi(J) \leq \phi(H)$ if and only if $J \leq H$, so that ϕ is one-to-one.

It remains to show that ϕ is onto. Given $\langle T_i \rangle_{i \in I} \in \prod \mathcal{I}(\mathbf{K}_i)$, let $J = \{x \in \sum L_i : x_i \in T_i \text{ for all } i\}$. Then $J \in \mathcal{I}(\sum \mathbf{K}_i)$, and it is not hard to see that $J_i = T_i$ for all i, and hence $\phi(J) = \langle T_i \rangle_{i \in I}$, as desired. □

Thus if $\mathcal{L}$ is a geometric lattice and $\mathbf{K} = \sum \mathbf{K}_i$, with each $\mathbf{K}_i$ simple, its ideal of compact elements, then $\mathcal{L} \cong \prod \mathcal{I}(\mathbf{K}_i)$. Now each $\mathbf{K}_i$ is a simple semimodular lattice in which every element is a finite join of atoms. The direct factors of $\mathcal{L}$ are ideal lattices of those types of lattices.

So consider an ideal lattice $\mathcal{H} = \mathcal{I}(\mathbf{K})$ where $\mathbf{K}$ is a simple semimodular lattice wherein every element is a join of finitely many atoms. We claim that $\mathcal{H}$ is subdirectly irreducible: the unique minimal congruence μ is generated by collapsing all the finite dimensional intervals of $\mathcal{H}$. This is because any two prime quotients in $\mathbf{K}$ are projective, which property is inherited by $\mathcal{H}$. So if $\mathbf{K}$ is finite dimensional, whence $\mathcal{H} \cong \mathbf{K}$, then $\mathcal{H}$ is simple, and it may be simple even though $\mathbf{K}$ is not finite dimensional, as is the case with $\mathbf{Eq}\, X$. On the other hand, if $\mathbf{K}$ is modular and infinite dimensional, then μ will identify only those pairs (a, b) such that $[a \wedge b, a \vee b]$ is finite dimensional, and so $\mathcal{L}$ will not be simple. Summarizing, we have the following result.

Theorem 11.5. *Every geometric lattice is a direct product of subdirectly irreducible geometric lattices. Every finite dimensional geometric lattice is a direct product of simple geometric lattices.*

The finite dimensional case of Theorem 11.5 should be credited to Dilworth [4], and the extension is due to J. Hashimoto [8]. The best version of Hashimoto's theorem states that *a complete, weakly atomic, relatively complemented lattice is a direct product of subdirectly irreducible lattices.* A nice variation, due to L. Libkin [10], is that *every atomistic algebraic lattice is a direct product of directly indecomposable (atomistic algebraic) lattices.*

Before going on to modular geometric lattices, we should mention one of the most intriguing problems in combinatorial lattice theory. Let $\mathcal{L}$ be a finite geometric lattice, and let

$$w_k = |\{x \in L : \delta(x) = k\}|.$$

The *unimodal conjecture* states that there is always an integer m such that

$$1 = w_0 \leq w_1 \leq \dots w_{m-1} \leq w_m \geq w_{m+1} \geq \dots w_{n-1} \geq w_n = 1.$$

This is true if $\mathcal{L}$ is modular, and also for $\mathcal{L} = \mathbf{Eq}\, X$ with X finite ([7] and [9]). It is known that $w_1 \leq w_k$ always holds for for $1 \leq k < n$ ([2] and [6]). But a general resolution of the conjecture still seems to be a long way off. For related results, see Dowling and Wilson [5].

We note in closing that a very different kind of geometry is obtained if one considers instead closure operators with the *anti-exchange property*: $y \in \Gamma(B \cup \{x\})$ and $y \notin \Gamma(B)$ implies $x \notin \Gamma(B \cup \{y\})$. For a comprehensive account of these convex geometries, as well as the appropriate history and original sources, see Adaricheva, Gorbunov and Tumanov [1].

Exercises for Chapter 11

1. This exercise gives a method for constructing new geometric lattices from known ones.

(a) Let $\mathcal{L}$ be a lattice, and let F be a nonempty order filter on $\mathcal{L}$, i.e., $x \geq f \in F$ implies $x \in F$. Show that the ordered set $\widehat{L}$ obtained by identifying all the elements of F (a join semilattice congruence) is a lattice.

(b) Let $\mathcal{L}$ be a finite dimensional semimodular lattice, with dimension n say, so that $\delta(1) = n$. Let $k < n$, and let the order filter F consist of all elements $x \in L$ with dimension $\delta(x) \geq k$. Show that $\widehat{L}$ is again semimodular, with dimension k. Note that if $\mathcal{L}$ is atomistic, then so is $\widehat{L}$.

(c) Give an example of a geometric lattice $\mathcal{L}$ and a filter F such that the lattice $\widehat{L}$ obtained by this construction is not semimodular.

2. Draw the following geometric lattices and their corresponding geometries:

(a) **Eq 4**,
(b) **Sub** $(Z_2)^3$, the lattice of subspaces of a 3-dimensional vector space over Z_2.

3. Show that each of the following is an algebraic closure operator on $\Re^n$, and interpret them geometrically. Which ones have the exchange property, and which the anti-exchange property?

(a) $\text{Span}(A) = \{\sum_{i=1}^{k} \lambda_i a_i : k \geq 1,\ a_i \in A \cup \{0\}\}$
(b) $\Gamma(A) = \{\sum_{i=1}^{k} \lambda_i a_i : k \geq 1,\ a_i \in A,\ \sum_{i=1}^{k} \lambda_i = 1\}$
(c) $\Delta(A) = \{\sum_{i=1}^{k} \lambda_i a_i : k \geq 1,\ a_i \in A,\ \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0\}$

4. Let G be a simple graph (no loops or multiple edges), and let X be the set of all edges of G. Define $S \subseteq X$ to be *closed* if whenever S contains all but one edge of a cycle, then it contains the entire cycle. Verify that the corresponding closure operator E is an algebraic closure operator with the exchange property. The lattice of E-closed subsets is called the *edge lattice* of G. Find the edge lattices of the graphs in Figure 11.1.

5. Show that the lattice for plane Euclidean geometry ($\Re^2$) is not modular. (Hint: Use two parallel lines and a point on one of them.)

6. (a) Let P and L be nonempty sets, which we will think of as "points" and "lines" respectively. Suppose we are given an arbitrary incidence relation $\in$ on $P \times L$. Then we can make $P \cup L \cup \{0, 1\}$ into a partially ordered set $\mathcal{K}$ in the obvious way,

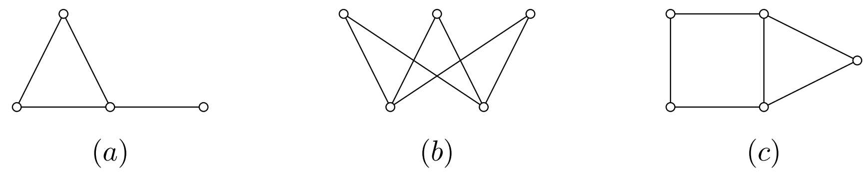

FIGURE 11.1

interpreting $p \in l$ as $p \le l$. When is $\mathcal{K}$ a lattice? atomistic? semimodular? modular? subdirectly irreducible?

(b) Compare these results with Hilbert's axioms for a plane geometry.

(i) There exists at least one line.
(ii) On each line there exist at least two points.
(iii) Not all points are on the same line.
(iv) There is one and only one line passing through two given distinct points.

7. Let $\mathcal{L}$ be a geometric lattice, and let A denote the set of atoms of $\mathcal{L}$. A subset $S \subseteq A$ is *independent* if $p \not\le \bigvee(S - \{p\})$ for all $p \in S$. A subset $B \subseteq A$ is a *basis* for $\mathcal{L}$ if B is independent and $\bigvee B = 1$.

(a) Prove that $\mathcal{L}$ has a basis.
(b) Prove that if B and C are bases for $\mathcal{L}$, then $|B| = |C|$.
(c) Show that the sublattice generated by an independent set S is isomorphic to the lattice of all finite subsets of S.

8. A lattice is *atomic* if for every $x > 0$ there exists $a \in L$ with $x \ge a \succ 0$. Prove that every element of a complete, relatively complemented, atomic lattice is a join of atoms.

9. Let I be an infinite set, and let $X = \{p_i : i \in I\} \dot\cup \{q_i : i \in I\}$. Define a subset S of X to be closed if $S = X$ or, for all i, at most one of p_i, q_i is in S. Let $\mathcal{L}$ be the lattice of all closed subsets of X.

(a) Prove that $\mathcal{L}$ is a relatively complemented algebraic lattice with every element the join of atoms.
(b) Show that the compact elements of $\mathcal{L}$ do not form an ideal.

(This example shows that the semimodularity hypothesis of Theorem 11.1 cannot be omitted.)

10. Prove that $\mathbf{Eq}\, X$ is relatively complemented and simple (Ore [13]).

11. Let $\mathcal{L}$ be a modular geometric lattice. Prove that $\mathcal{L}$ is subdirectly irreducible (in the finite dimensional case, simple) if and only if the following condition holds: *for any two distinct atoms a, b of $\mathcal{L}$, there exists a third atom c such that* $a \vee b = a \vee c = b \vee c$, i.e., the three atoms generate a diamond. Give an example to show that this condition is not necessary in the nonmodular (but still semimodular) case.

12. On a modular lattice $\mathcal{M}$, define a relation $a \, \mu \, b$ iff $[a \wedge b, a \vee b]$ has finite length. Show that μ is a congruence relation.

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