7. Varieties of Lattices

Variety is the spice of life.

A *lattice equation* is an expression $p \approx q$ where p and q are lattice terms. Our intuitive notion of what it means for a lattice $\mathcal{L}$ to satisfy $p \approx q$ is that $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ whenever elements of $\mathcal{L}$ are substituted for the variables. This is captured by the formal definition: $\mathcal{L}$ *satisfies* $p \approx q$ if $h(p) = h(q)$ for every homomorphism $h : W(X) \to \mathcal{L}$. We say that $\mathcal{L}$ satisfies a set Σ of equations if $\mathcal{L}$ satisfies every equation in Σ. Likewise, a class $\mathcal{K}$ of lattices satisfies Σ if every lattice $\mathcal{L} \in \mathcal{K}$ does so.

As long as we are dealing entirely with lattices, there is no loss of generality in replacing p and q by the corresponding elements of $\mathrm{FL}(X)$, since if terms p and p' evaluate the same in $\mathrm{FL}(X)$, then they evaluate the same for every substitution in every lattice. In practice it is often more simple and natural to think of equations between elements in a free lattice, rather than the corresponding terms, as in Theorem 7.2 below.

A *variety* (or *equational class*) of lattices is the class of all lattices satisfying some set Σ of lattice equations. The class $\mathbf{L}$ of all lattices is defined by equations (the idempotent, commutative, associative and absorption laws), so it forms a variety. Contained within $\mathbf{L}$ are some familiar subvarieties:

(1) the variety $\mathbf{M}$ of modular lattices, satisfying $(x \vee y) \wedge (x \vee z) \approx x \vee (z \wedge (x \vee y))$;
(2) the variety $\mathbf{D}$ of distributive lattices, satisfying $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;
(3) the variety $\mathbf{T}$ of one-element lattices, satisfying $x \approx y$ (not very exciting).

If $\mathbf{K}$ is any class of lattices, we say that a lattice $\mathcal{F}$ is $\mathbf{K}$-*freely generated* by its subset X if

(1) $\mathcal{F} \in \mathbf{K}$,
(2) X generates $\mathcal{F}$,
(3) for every lattice $\mathcal{L} \in \mathbf{K}$, every map $h_0 : X \to L$ can be extended to a homomorphism $h : \mathcal{F} \to \mathcal{L}$.

A lattice is $\mathbf{K}$-*free* if it is $\mathbf{K}$-freely generated by one of its subsets, and *relatively free* if it is $\mathbf{K}$-free for some (unspecified) class $\mathbf{K}$.

While these ideas floated around for some time before, it was Garrett Birkhoff [5] who proved the basic theorem about varieties in the 1930's.

Theorem 7.1. *If* $\mathbf{K}$ *is a nonempty class of lattices, then the following are equivalent.*

(1) $\mathbf{K}$ *is a variety.*

(2) $\mathbf{K}$ *is closed under the formation of homomorphic images, sublattices and direct products.*
(3) *Either* $\mathbf{K} = \mathbf{T}$ *(the variety of one-element lattices), or for every nonempty set* X *there is a lattice* $\mathcal{F}_{\mathbf{K}}(X)$ *that is* $\mathbf{K}$*-freely generated by* X*, and* $\mathbf{K}$ *is closed under homomorphic images.*

Proof. It is easy to see that varieties are closed under homomorphic images, sublattices and direct products, so (1) implies (2).

The crucial step in the equivalence, the construction of relatively free lattices $\mathcal{F}_{\mathbf{K}}(X)$, is a straightforward adaptation of the construction of $\mathrm{FL}(X)$. Let $\mathbf{K}$ be a class that is closed under the formation of sublattices and direct products, and let $\kappa = \bigcap\{\theta \in \mathbf{Con}\, W(X) : W(X)/\theta \in \mathbf{K}\}$. Following the proof of Theorem 6.1, we can show that $W(X)/\kappa$ is a subdirect product of lattices in $\mathbf{K}$, and that it is $\mathbf{K}$-freely generated by $\{x\kappa : x \in X\}$. Unless $\mathbf{K} = \mathbf{T}$, the classes $x\kappa$ $(x \in X)$ will be distinct. Thus (2) implies (3).

Finally, suppose that $\mathbf{K}$ is a class of lattices that is closed under homomorphic images and contains a $\mathbf{K}$-freely generated lattice $\mathcal{F}_{\mathbf{K}}(X)$ for every nonempty set X. For each nonempty X there is a homomorphism $f_X : W(X) \twoheadrightarrow \mathcal{F}_{\mathbf{K}}(X)$ that is the identity on X. Fix the countably infinite set $X_0 = \{x_1, x_2, x_3, \dots\}$, and let Σ be the collection of all equations $p \approx q$ such that $(p, q) \in \ker f_{X_0}$. Thus $p \approx q$ is in Σ if and only if $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$ in the countably generated lattice $\mathcal{F}_{\mathbf{K}}(X_0) \cong \mathcal{F}_{\mathbf{K}}(\omega)$.

Let $\mathbf{V}_\Sigma$ be the variety of all lattices satisfying Σ; we want to show that $\mathbf{K} = \mathbf{V}_\Sigma$. We formulate the critical argument as a sublemma.

Sublemma. *Let* $\mathcal{F}_{\mathbf{K}}(Y)$ *be a relatively free lattice. Let* $p, q \in W(Y)$ *and let* $f_Y : W(Y) \twoheadrightarrow \mathcal{F}_{\mathbf{K}}(Y)$ *with* f_Y *the identity on* Y*. Then* $\mathbf{K}$ *satisfies* $p \approx q$ *if and only if* $f_Y(p) = f_Y(q)$*.*

Proof. If $\mathbf{K}$ satisfies $p \approx q$, then $f_Y(p) = f_Y(q)$ because $\mathcal{F}_{\mathbf{K}}(Y) \in \mathbf{K}$. Conversely, if $f_Y(p) = f_Y(q)$, then by the mapping property (III) every lattice in $\mathbf{K}$ satisfies $p \approx q$.[1] □

Applying the Sublemma with $Y = X_0$, we conclude that $\mathbf{K}$ satisfies every equation of Σ, so $\mathbf{K} \subseteq \mathbf{V}_\Sigma$.

Conversely, let $\mathcal{L} \in \mathbf{V}_\Sigma$, and let X be a generating set for $\mathcal{L}$. The identity map on X extends to a surjective homomorphism $h : W(X) \twoheadrightarrow \mathcal{L}$, and we also have the map $f_X : W(X) \to \mathcal{F}_{\mathbf{K}}(X)$. For any pair $(p, q) \in \ker f_X$, the Sublemma says that $\mathbf{K}$ satisfies $p \approx q$. Again by the Sublemma, there is a corresponding equation in Σ (perhaps involving different variables). Since $\mathcal{L} \in \mathbf{V}_\Sigma$ this implies $h(p) = h(q)$. So $\ker f_X \leq \ker h$, and hence by the Second Isomorphism Theorem there is

[1] However, if Y is finite and $Y \subseteq Z$, then $\mathcal{F}_{\mathbf{K}}(Y)$ may satisfy equations not satisfied by $\mathcal{F}_{\mathbf{K}}(Z)$. For example, for any lattice variety, $\mathcal{F}_{\mathbf{K}}(2)$ is distributive. The Sublemma only applies to equations with at most $|Y|$ variables.

a homomorphism $g : \mathcal{F}_{\mathbf{K}}(X) \to \mathcal{L}$ such that $h = gf_X$. Thus $\mathcal{L}$ is a homomorphic image of $\mathcal{F}_{\mathbf{K}}(X)$. Since $\mathbf{K}$ is closed under homomorphic images, this implies $\mathcal{L} \in \mathbf{K}$. Hence $\mathbf{V}_\Sigma \subseteq \mathbf{K}$, and equality follows. Therefore (3) implies (1). $\square$

The three parts of Theorem 7.1 reflect three different ways of looking at varieties. The first is to start with a set Σ of equations, and to consider the variety $V(\Sigma)$ of all lattices satisfying those equations. The given equations will in general imply other equations, *viz.*, all the relations holding in the relatively free lattices $\mathcal{F}_{V(\Sigma)}(X)$. It is important to notice that while the proof of Birkhoff's theorem tells us abstractly how to construct relatively free lattices, it does not tell us how to solve the word problem for them. Consider the variety $\mathbf{M}$ of modular lattices. Richard Dedekind [6] showed in the 1890's that $\mathcal{F}_{\mathbf{M}}(3)$ has 28 elements; it is drawn in Figure 9.2. On the other hand, Ralph Freese [9] proved in 1980 that the word problem for $\mathcal{F}_{\mathbf{M}}(5)$ is unsolvable: *there is no algorithm for determining whether* $p = q$ *in* $\mathcal{F}_{\mathbf{M}}(5)$. Christian Herrmann [10] later showed that the word problem for $\mathcal{F}_{\mathbf{M}}(4)$ is also unsolvable. It follows, by the way, that the variety of modular lattices is not generated by its finite members:[2] *there is a lattice equation that holds in all finite modular lattices, but not in all modular lattices.*

Skipping to the third statement of Theorem 7.1, let $\mathbf{V}$ be a variety, and let κ be the kernel of the natural homomorphism $h : \mathrm{FL}(X) \to \mathcal{F}_{\mathbf{V}}(X)$ with $h(x) = x$ for all $x \in X$. Then, of course, $\mathcal{F}_{\mathbf{V}}(X) \cong \mathrm{FL}(X)/\kappa$. We want to ask which congruences on $\mathrm{FL}(X)$ arise in this way, i.e., *for which* $\theta \in \mathbf{Con}\ \mathrm{FL}(X)$ *is* $\mathrm{FL}(X)/\theta$ *relatively free?* To answer this, we need a couple of definitions.

An *endomorphism* of a lattice $\mathcal{L}$ is a homomorphism $f : \mathcal{L} \to \mathcal{L}$. The set of endomorphisms of $\mathcal{L}$ forms a semigroup $\mathbf{End}\ \mathcal{L}$ under composition. It is worth noting that an endomorphism of a lattice is determined by its action on a generating set, since $f(p(x_1, \ldots, x_n) = p(f(x_1), \ldots, f(x_n))$ for any lattice term p. In particular, an endomorphism f of $\mathrm{FL}(X)$ corresponds to a substitution $x_i \mapsto f(x_i)$ of elements for the generators.

A congruence relation θ is *fully invariant* if $(x, y) \in \theta$ implies $(f(x), f(y)) \in \theta$ for every endomorphism f of $\mathcal{L}$. The fully invariant congruences of $\mathcal{L}$ can be thought of as the congruence relations of the algebra $\mathcal{L}^* = (L, \wedge, \vee, \{f : f \in \mathrm{End}\ \mathcal{L}\})$. In particular, they form an algebraic lattice, in fact a complete sublattice of $\mathbf{Con}\,\mathcal{L}$.

The answer to our question, in these terms, is again due to Garrett Birkhoff [4].

Theorem 7.2. $\mathrm{FL}(X)/\theta$ *is relatively freely generated by* $\{x\theta : x \in X\}$ *if and only if* θ *is fully invariant.*

Proof. Let $\mathbf{V}$ be a lattice variety and let $h : \mathrm{FL}(X) \to \mathcal{F}_{\mathbf{V}}(X)$ with $h(x) = x$ for all $x \in X$. Then $h(p) = h(q)$ if and only if $\mathbf{V}$ satisfies $p \approx q$ (as in the Sublemma).

[2]If a variety $\mathbf{V}$ of algebras (1) has only finitely many operation symbols, (2) is finitely based, and (3) is generated by its finite members, then the word problem for $\mathcal{F}_{\mathbf{V}}(X)$ is solvable. This result is due to A. I. Malcev for groups; see T. Evans [7].

Hence, for any endomorphism f and elements p, $q \in \mathrm{FL}(X)$, if $h(p) = h(q)$ then

$$\begin{aligned} hf(p) = h(f(p(x_1,\dots,x_n))) &= h(p(f(x_1),\dots,f(x_n))) \\ &= h(q(f(x_1),\dots,f(x_n))) \\ &= h(f(q(x_1,\dots,x_n))) = hf(q) \end{aligned}$$

so that $(f(p), f(q)) \in \ker h$. Thus the congruence $\ker h$ is fully invariant.

Conversely, assume that θ is a fully invariant congruence on $\mathrm{FL}(X)$. If $\theta = 1_{\mathbf{Con}\ \mathrm{FL}(X)}$, then θ is fully invariant and $\mathrm{FL}(X)/\theta$ is relatively free for the trivial variety $\mathbf{T}$. So without loss of generality, θ is not the universal relation. Let $k : \mathrm{FL}(X) \to \mathrm{FL}(X)/\theta$ be the canonical homomorphism with $\ker k = \theta$. Let $\mathbf{V}$ be the variety determined by the set of equations $\Sigma = \{p \approx q : (p, q) \in \theta\}$. To show that $\mathrm{FL}(X)/\theta$ is $\mathbf{V}$-freely generated by $\{x\theta : x \in X\}$, we must verify that

(1) $\mathrm{FL}(X)/\theta \in \mathbf{V}$, and
(2) if $\mathcal{M} \in \mathbf{V}$ and $h_0 : X \to M$, then there is a homomorphism $h : \mathrm{FL}(X)/\theta \to \mathcal{M}$ such that $h(x\theta) = h_0(x)$, i.e., $hk(x) = h_0(x)$ for all $x \in X$.

For (1), we must show that the lattice $\mathrm{FL}(X)/\theta$ satisfies every equation of Σ, i.e., that if $p(x_1,\dots,x_n)\ \theta\ q(x_1,\dots,x_n)$ and $w_1,\dots,w_n$ are elements of $\mathrm{FL}(X)$, then $p(w_1,\dots,w_n)\ \theta\ q(w_1,\dots,w_n)$. Since there is an endomorphism f of $\mathrm{FL}(X)$ with $f(x_i) = w_i$ for all i, this follows from the fact that θ is fully invariant.

To prove (2), let $g : \mathrm{FL}(X) \to \mathcal{M}$ be the homomorphism such that $g(x) = h_0(x)$ for all $x \in X$. Since $\mathcal{M}$ is in $\mathbf{V}$, $g(p) = g(q)$ whenever $p \approx q$ is in Σ, and thus $\theta = \ker k \le \ker g$. By the Second Isomorphism Theorem, there is a homomorphism $h : \mathrm{FL}(X)/\theta \to \mathcal{M}$ such that $hk = g$, as desired. □

It follows that varieties of lattices are in one-to-one correspondence with fully invariant congruences on $\mathrm{FL}(\omega)$. The consequences of this fact can be summarized as follows.

Theorem 7.3. *The set of all lattice varieties ordered by containment forms a lattice Λ that is dually isomorphic to the lattice of all fully invariant congruences of* $\mathrm{FL}(\omega)$. *Thus Λ is dually algebraic, and a variety $\mathbf{V}$ is dually compact in Λ if and only if $\mathbf{V} = V(\Sigma)$ for some finite set of equations Σ.*

Going back to statement (2) of Theorem 7.1, the third way of looking at varieties is model theoretic: a variety is a class of lattices closed under the operators H (homomorphic images), S (sublattices) and P (direct products). Now elementary arguments show that, for any class $\mathbf{K}$,

$$\begin{aligned} \mathrm{PS}(\mathbf{K}) &\subseteq \mathrm{SP}(\mathbf{K}) \\ \mathrm{PH}(\mathbf{K}) &\subseteq \mathrm{HP}(\mathbf{K}) \\ \mathrm{SH}(\mathbf{K}) &\subseteq \mathrm{HS}(\mathbf{K}). \end{aligned}$$

Thus the smallest variety containing a class $\mathbf{K}$ of lattices is $\mathrm{HSP}(\mathbf{K})$, the class of all homomorphic images of sublattices of direct products of lattices in $\mathbf{K}$. We refer to $\mathrm{HSP}(\mathbf{K})$ as the variety *generated by* $\mathbf{K}$. We can think of HSP as a closure operator, but not an algebraic one: Λ is not upper continuous, so it cannot be algebraic (see Exercise 6). The many advantages of this point of view will soon become apparent.

Lemma 7.4. *Two lattice varieties are equal if and only if they contain the same subdirectly irreducible lattices.*

Proof. Recall from Theorem 5.6 that every lattice $\mathcal{L}$ is a subdirect product of subdirectly irreducible lattices $\mathcal{L}/\varphi$ with φ completely meet irreducible in $\mathbf{Con}\ \mathcal{L}$. Suppose $\mathbf{V}$ and $\mathbf{K}$ are varieties, and that the subdirectly irreducible lattices of $\mathbf{V}$ are all in $\mathbf{K}$. Then for any X the relatively free lattice $\mathcal{F}_{\mathbf{V}}(X)$, being a subdirect product of subdirectly irreducible lattices $\mathcal{F}_{\mathbf{V}}(X)/\varphi$ in $\mathbf{V}$, is a subdirect product of lattices in $\mathbf{K}$. Hence $\mathcal{F}_{\mathbf{V}}(X) \in \mathbf{K}$ and $\mathbf{V} \subseteq \mathbf{K}$. The lemma follows by symmetry. □

This leads us directly to a crucial question: *If* $\mathbf{K}$ *is a set of lattices, how can we find the subdirectly irreducible lattices in* $\mathrm{HSP}(\mathbf{K})$*?* The answer, due to Bjarni Jónsson, requires that we once again venture into the world of logic.

Let us recall that a *filter* (or *dual ideal*) of a lattice $\mathcal{L}$ with greatest element 1 is a subset F of L such that

(1) $1 \in F$,
(2) $x, y \in F$ implies $x \wedge y \in F$,
(3) $z \geq x \in F$ implies $z \in F$.

For any $x \in L$, the set $\uparrow x$ is called a *principal filter*. As an example of a nonprincipal filter, in the lattice $\mathfrak{P}(X)$ of all subsets of an infinite set X we have the filter F of all complements of finite subsets of X. A maximal proper filter is called an *ultrafilter*.

We want to describe an important type of congruence relation on direct products. Let $\mathcal{L}_i$ $(i \in I)$ be lattices, and let F be a filter on the lattice of subsets $\mathfrak{P}(I)$. We define an equivalence relation $\equiv_F$ on the direct product $\prod_{i \in I} \mathcal{L}_i$ by

$$x \equiv_F y \text{ if } \{i \in I : x_i = y_i\} \in F.$$

A routine check shows that $\equiv_F$ is a congruence relation.

Lemma 7.5. *(1) Let $\mathcal{L}$ be a lattice, F a filter on $\mathcal{L}$, and $a \notin F$. Then there exists a filter G on $\mathcal{L}$ maximal with respect to the properties $F \subseteq G$ and $a \notin G$.*

(2) A proper filter U on $\mathfrak{P}(I)$ is an ultrafilter if and only if for every $A \subseteq I$, either $A \in U$ or $I - A \in U$.

(3) If U is an ultrafilter on $\mathfrak{P}(I)$, then its complement $\mathfrak{P}(I) - U$ is a maximal proper ideal.

(4) If U is an ultrafilter and $A_1 \cup \cdots \cup A_n \in U$, then $A_i \in U$ for some i.

(5) An ultrafilter U is nonprincipal if and only if it contains the filter of all complements of finite subsets of I.

Proof. Part (1) is a straightforward Zorn's Lemma argument. Moreover, it is clear that a proper filter U is maximal if and only if for every $A \notin U$ there exists $B \in U$ such that $A \cap B = \emptyset$, i.e., $B \subseteq I - A$. For if F is a filter and A is a subset of I with the property that $A \notin F$ and $A \cap B \neq \emptyset$ for all $B \in F$, then the filter G generated by $F \cup \{A\}$ does not contain $\emptyset$, and $G \supset F$ properly. Thus U is an ultrafilter if and only if $A \notin U$ implies $I - A \in U$, which is (2). DeMorgan's Laws then yield (3), which in turn implies (4). It follows from (4) that if an ultrafilter U on I contains a finite set, then it contains a singleton $\{i_0\}$, and hence is principal with $U = \uparrow\{i_0\} = \{A \subseteq I : i_0 \in A\}$. Conversely, if U is a principal ultrafilter $\uparrow S$, then S must be a singleton. Thus an ultrafilter is nonprincipal if and only if it contains no finite set, which by (2) means that it contains the complement of every finite set. □

Corollary. *If I is an infinite set, then there is a nonprincipal ultrafilter on $\mathfrak{P}(I)$.*

Proof. Apply Lemma 7.5(1) with $\mathcal{L} = \mathfrak{P}(I)$, F the filter of all complements of finite subsets of I, and $a = \emptyset$. □

If F is a filter on $\mathfrak{P}(I)$, the quotient lattice $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{F}}$ is called a *reduced product*. If U is an ultrafilter, then $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{U}}$ is an *ultraproduct*. The interesting case is when U is a nonprincipal ultrafilter. Good references on reduced products and ultraproducts are [3] and [8].

Our next immediate goal is to investigate what properties are preserved by the ultraproduct construction. In order to be precise, we begin with a slough of definitions, reserving comment for later.

The elements of a *first order language* for lattices are

(1) a countable alphabet X with members denoted $x, y, z, \ldots$,
(2) equations $p \approx q$ with $p, q \in W(X)$,
(3) logical connectives AND, OR, and $\neg$,
(4) quantifiers $\forall x$ and $\exists x$ for all $x \in X$.

These symbols can be combined appropriately to form *well formed formulas* (wffs) by the following rules.

(1) Every equation $p \approx q$ is a wff.
(2) If α and β are wffs, then so are $(\neg\alpha)$, $(\alpha$ AND $\beta)$ and $(\alpha$ OR $\beta)$.
(3) If γ is a wff and $x \in X$, then $(\forall x \gamma)$ and $(\exists x \gamma)$ are wffs.
(4) Only expressions generated by the first three rules are wffs.

Now let $\mathcal{L}$ be a lattice, let $h : W(X) \to \mathcal{L}$ be a homomorphism, and let φ be a well formed formula. We say that the pair $(\mathcal{L}, h)$ *models* φ, written symbolically as $(\mathcal{L}, h) \models \varphi$, according to the following recursive definition. By way of notation, for $g : W(X) \to \mathcal{L}$ and $Y \subseteq X$, $g|_Y$ denotes the restriction of g to Y.

(1) $(\mathcal{L}, h) \models p \approx q$ if $h(p) = h(q)$, i.e., if $p(h(x_1), \ldots, h(x_n)) = q(h(x_1), \ldots, h(x_n))$.
(2) $(\mathcal{L}, h) \models (\neg\alpha)$ if $(\mathcal{L}, h)$ does not model α (written $(\mathcal{L}, h) \not\models \alpha$).
(3) $(\mathcal{L}, h) \models (\alpha$ AND $\beta)$ if $(\mathcal{L}, h) \models \alpha$ and $(\mathcal{L}, h) \models \beta$.

(4) $(\mathcal{L}, h) \models (\alpha \text{ OR } \beta)$ if $(\mathcal{L}, h) \models \alpha$ or $(\mathcal{L}, h) \models \beta$ (or both).
(5) $(\mathcal{L}, h) \models (\forall x \gamma)$ if $(\mathcal{L}, g) \models \gamma$ for every g such that $g|_{X-\{x\}} = h|_{X-\{x\}}$.
(6) $(\mathcal{L}, h) \models (\exists x \gamma)$ if $(\mathcal{L}, g) \models \gamma$ for some g such that $g|_{X-\{x\}} = h|_{X-\{x\}}$.

Finally, $\mathcal{L}$ *satisfies* φ if $(\mathcal{L}, h)$ models φ for every homomorphism $h : W(X) \to \mathcal{L}$.

We are particularly interested in well formed formulas φ for which all the variables appearing in φ are quantified (by $\forall$ or $\exists$). The set F_φ of variables that *occur freely in* φ is defined recursively as follows.

(1) For an equation, $F_{p \approx q}$ is the set of all variables that actually appear in p or q.
(2) $F_{\neg\alpha} = F_\alpha$.
(3) $F_{\alpha \text{ AND } \beta} = F_\alpha \cup F_\beta$.
(4) $F_{\alpha \text{ OR } \beta} = F_\alpha \cup F_\beta$.
(5) $F_{\forall x \alpha} = F_\alpha - \{x\}$.
(6) $F_{\exists x \alpha} = F_\alpha - \{x\}$.

A *first order sentence* is a well formed formula φ such that F_φ is empty, i.e., no variable occurs freely in φ. It is not hard to show inductively that, for a given lattice $\mathcal{L}$ and any well formed formula φ, whether or not $(\mathcal{L}, h) \models \varphi$ is true depends only on the values of $h|_{F_\varphi}$, i.e., if $g|_{F_\varphi} = h|_{F_\varphi}$, then $(\mathcal{L}, g) \models \varphi$ iff $(\mathcal{L}, h) \models \varphi$. So if φ is a sentence, then either $\mathcal{L}$ satisfies φ or $\mathcal{L}$ satisfies $\neg\varphi$.

Now some comments are in order. First of all, we did not include the predicate $p \le q$ because we can capture it with the equation $p \vee q \approx q$. Likewise, the logical connective $\Longrightarrow$ is omitted because $(\alpha \Longrightarrow \beta)$ is equivalent to $(\neg\alpha)$ OR β. On the other hand, our language is redundant because OR can be eliminated by the use of DeMorgan's law, and $\exists x \varphi$ is equivalent to $\neg\forall x(\neg\varphi)$.

Secondly, for any well formed formula φ, a lattice $\mathcal{L}$ satisfies φ if and only if it satisfies the sentence $\forall x_{i_1} \dots \forall x_{i_k} \varphi$ where the quantification runs over the variables in F_φ. Thus we can consistently speak of a lattice satisfying an equation or Whitman's condition, for example, when what we really have in mind is the corresponding universally quantified sentence.

Fortunately, our intuition about what sort of properties can be expressed as first order sentences, and what it means for a lattice to satisfy a sentence φ, tends to be pretty good, particularly after we have seen a lot of examples. With this in mind, let us list some first order properties.

(1) $\mathcal{L}$ satisfies $p \approx q$.
(2) $\mathcal{L}$ satisfies the semidistributive laws $(SD_\vee)$ and $(SD_\wedge)$.
(3) $\mathcal{L}$ satisfies Whitman's condition (W).
(4) $\mathcal{L}$ has width 7.
(5) $\mathcal{L}$ has at most 7 elements.
(6) $\mathcal{L}$ has exactly 7 elements.
(7) $\mathcal{L}$ is isomorphic to $\mathcal{M}_5$.

And, of course, we can do negations and finite conjunctions and disjunctions of these. The sort of things that *cannot* be expressed by first order sentences includes the following.

(1) $\mathcal{L}$ is finite.
(2) $\mathcal{L}$ satisfies the ACC.
(3) $\mathcal{L}$ has finite width.
(4) $\mathcal{L}$ is subdirectly irreducible.

Now we are in a position to state for lattices the fundamental theorem about ultraproducts, due to J. Łos in 1955 [14].

Theorem 7.6. *Let φ be a first order lattice sentence, $\mathcal{L}_i$ $(i \in I)$ lattices, and U an ultrafilter on $\mathfrak{P}(I)$. Then the ultraproduct $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{U}}$ satisfies φ if and only if $\{i \in I : \mathcal{L}_i$ satisfies $\varphi\}$ is in U.*

Corollary. *If each $\mathcal{L}_i$ satisfies φ, then so does the ultraproduct $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{U}}$.*

Proof. Suppose we have a collection of lattices $\mathcal{L}_i$ $(i \in I)$ and an ultrafilter U on $\mathfrak{P}(I)$. The elements of the ultraproduct $\prod_{i \in I} \mathcal{L}_i / \equiv_{\mathrm{U}}$ are equivalence classes of elements of the direct product. Let $\mu : \prod \mathcal{L}_i \to \prod \mathcal{L}_i / \equiv_{\mathrm{U}}$ be the canonical homomorphism, and let $\pi_j : \prod \mathcal{L}_i \to \mathcal{L}_j$ denote the projection map. We will prove the following claim, which includes Theorem 7.6.

Claim. *Let $h : W(X) \to \prod_{i \in I} \mathcal{L}_i$ be a homomorphism, and let φ be a well formed formula. Then $(\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \varphi$ if and only if $\{i \in I : (\mathcal{L}_i, \pi_i h) \models \varphi\} \in U$.*

We proceed by induction on the complexity of φ. In view of the observations above (e.g., DeMorgan's Laws), it suffices to treat equations, AND, $\neg$ and $\forall$. The first three are quite straightforward.

Note that for $a, b \in \prod \mathcal{L}_i$ we have $\mu(a) = \mu(b)$ if and only if $\{i : \pi_i(a) = \pi_i(b)\} \in U$. Thus, for an equation $p \approx q$, we have

$$\begin{aligned} (\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models p \approx q \quad &\text{iff} \quad \mu h(p) = \mu h(q) \\ &\text{iff} \quad \{i : \pi_i h(p) = \pi_i h(q)\} \in U \\ &\text{iff} \quad \{i : (\mathcal{L}_i, \pi_i h) \models p \approx q\} \in U. \end{aligned}$$

.

For a conjunction α AND β, using $A \cap B \in U$ iff $A \in U$ and $B \in U$, we have

$$\begin{aligned} (\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \alpha \text{ AND } \beta \text{ iff } &(\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \alpha \text{ and } (\textstyle\prod \mathcal{L}_i / \equiv_{\mathrm{U}}, \mu h) \models \beta \\ \text{iff } &\{i : (\mathcal{L}_i, \pi_i h) \models \alpha\} \in U \text{ and } \{i : (\mathcal{L}_i, \pi_i h) \models \beta\} \in U \\ \text{iff } &\{i : (\mathcal{L}_i, \pi_i h) \models \alpha \text{ AND } \beta\} \in U. \end{aligned}$$

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For a negation $\neg\alpha$, using the fact that $A \in U$ iff $I - A \notin U$, we have

$$\begin{aligned}
(\prod \mathcal{L}_i/\equiv_{\mathrm{U}}, \mu h) \models \neg\alpha \quad &\text{iff} \quad (\prod \mathcal{L}_i/\equiv_{\mathrm{U}}, \mu h) \not\models \alpha \\
&\text{iff} \quad \{i : (\mathcal{L}_i, \pi_i h) \models \alpha\} \notin U \\
&\text{iff} \quad \{j : (\mathcal{L}_j, \pi_j h) \not\models \alpha\} \in U \\
&\text{iff} \quad \{j : (\mathcal{L}_j, \pi_j h) \models \neg\alpha\} \in U.
\end{aligned}$$

.

Finally, we consider the case when φ has the form $\forall x\gamma$. First, assume $A = \{i : (\mathcal{L}_i, \pi_i h) \models \forall x\gamma\} \in U$, and let $g : W(X) \to \prod \mathcal{L}_i$ be a homomorphism such that $\mu g|_{X-\{x\}} = \mu h|_{X-\{x\}}$. This means that for each $y \in X - \{x\}$, the set $B_y = \{j : \pi_j g(y) = \pi_j h(y)\} \in U$. Since F_γ is a finite set and U is closed under intersection, it follows that $B = \bigcap_{y \in F_\gamma - \{x\}} B_y = \{j : \pi_j g(y) = \pi_j h(y) \text{ for all } y \in F_\gamma - \{x\}\} \in U$. Therefore $A \cap B = \{i : (\mathcal{L}_i, \pi_i h) \models \forall x\gamma \text{ and } \pi_i g|_{F_\gamma - \{x\}} = \pi_i h|_{F_\gamma - \{x\}}\} \in U$. Hence $\{i : (\mathcal{L}_i, \pi_i g) \models \gamma\} \in U$, and so by induction $(\prod \mathcal{L}_i/\equiv_{\mathrm{U}}, \mu g) \models \gamma$. Thus $(\prod \mathcal{L}_i/\equiv_{\mathrm{U}}, \mu h) \models \forall x\gamma$, as desired.

Conversely, suppose $A = \{i : (\mathcal{L}_i, \pi_i h) \models \forall x\gamma\} \notin U$. Then the complement $I - A = \{j : (\mathcal{L}_j, \pi_j h) \not\models \forall x\gamma\} \in U$. For each $j \in I - A$, there is a homomorphism $g_j : W(X) \to \mathcal{L}_j$ such that $g_j|_{X-\{x\}} = \pi_j h|_{X-\{x\}}$ and $(L_j, g_j) \not\models \gamma$. Let $g : W(X) \to \prod \mathcal{L}_i$ be a homomorphism such that $\pi_j g = g_j$ for all $j \in I - A$. Then $\mu g|_{X-\{x\}} = \mu h|_{X-\{x\}}$ but $(\prod \mathcal{L}_i/\equiv_{\mathrm{U}}, \mu g) \not\models \gamma$. Thus $(\prod \mathcal{L}_i/\equiv_{\mathrm{U}}, \mu h) \not\models \forall x\gamma$.

This completes the proof of Lemma 7.6. □

To our operators H, S and P let us add a fourth: $\mathrm{P_u}(\mathbf{K})$ is the class of all ultraproducts of lattices from $\mathbf{K}$. Finally we get to answer the question: *Where do subdirectly irreducibles come from?*

Theorem 7.7. Jónsson's Lemma. *Let $\mathbf{K}$ be a class of lattices. If $\mathcal{L}$ is subdirectly irreducible and $\mathcal{L} \in \mathrm{HSP}(\mathbf{K})$, then $\mathcal{L} \in \mathrm{HSP_u}(\mathbf{K})$.*

Proof. Now $\mathcal{L} \in \mathrm{HSP}(\mathbf{K})$ means that there are lattices $\mathcal{K}_i \in \mathbf{K}$ $(i \in I)$, a sublattice $\mathcal{S}$ of $\prod_{i \in I} \mathcal{K}_i$, and a surjective homomorphism $h : \mathcal{S} \twoheadrightarrow \mathcal{L}$. If we also assume that $\mathcal{L}$ is finitely subdirectly irreducible (this suffices), then $\ker h$ is meet irreducible in $\mathbf{Con}\,\mathcal{S}$. Since $\mathbf{Con}\,\mathcal{S}$ is distributive, this makes $\ker h$ meet prime, i.e., $\varphi \wedge \psi \leq \ker h$ implies $\varphi \leq \ker h$ or $\psi \leq \ker h$.

For any $J \subseteq I$, let π_J be the kernel of the projection of $\mathcal{S}$ onto $\prod_{j \in J} \mathcal{K}_j$. Thus for $a, b \in S$ we have $a \; \pi_J \; b$ iff $a_j = b_j$ for all $j \in J$. Note that $H \supseteq J$ implies $\pi_H \leq \pi_J$, and that $\pi_{J \cup K} = \pi_J \wedge \pi_K$.

Let $\mathfrak{H} = \{J \subseteq I : \pi_J \leq \ker h\}$. By the preceding observations,

(1) $I \in \mathfrak{H}$ and $\emptyset \notin \mathfrak{H}$,
(2) $\mathfrak{H}$ is an order filter in $\mathfrak{P}(I)$,
(3) $J \cup K \in \mathfrak{H}$ implies $J \in \mathfrak{H}$ or $K \in \mathfrak{H}$.

However, $\mathfrak{H}$ need not be a (lattice) filter. Let us therefore consider

$$\mathcal{Q} = \{F \subseteq \mathfrak{P}(I) : F \text{ is a filter on } \mathfrak{P}(I) \text{ and } F \subseteq \mathfrak{H}\}.$$

By Zorn's Lemma, $\mathcal{Q}$ contains a maximal member with respect to set inclusion, say U. Let us show that U is an ultrafilter.

If not, then by Lemma 7.5(2) there exists $A \subseteq I$ such that A and $I - A$ are both not in U. By the maximality of U, this means that there exists a subset $X \in U$ such that $A \cap X \notin \mathfrak{H}$. Similarly, there is a $Y \in U$ such that $(I - A) \cap Y \notin \mathfrak{H}$. Let $Z = X \cap Y$. Then $Z \in U$, and hence $Z \in \mathfrak{H}$. However, $A \cap Z \subseteq A \cap X$, whence $A \cap Z \notin \mathfrak{H}$ by (2) above. Likewise $(I - A) \cap Z \notin \mathfrak{H}$. But

$$(A \cap Z) \cup ((I - A) \cap Z) = Z \in \mathfrak{H},$$

contradicting (3). Thus U is an ultrafilter.

Now $\equiv_U \in \textbf{Con} \prod \mathcal{K}_i$, and its restriction is a congruence on $\mathcal{S}$. Moreover, $\mathcal{S}/\equiv_{\mathrm{U}}$ is (isomorphic to) a sublattice of $\prod \mathcal{K}_i/\equiv_{\mathrm{U}}$. If a, b are any pair of elements of $\mathcal{S}$ such that $a \equiv_U b$, then $J = \{i : a_i = b_i\} \in U$. This implies $J \in \mathfrak{H}$ and so $\pi_J \leq \ker h$, whence $h(a) = h(b)$. Thus the restriction of $\equiv_U$ to $\mathcal{S}$ is below $\ker h$, wherefore $\mathcal{L} = h(\mathcal{S})$ is a homomorphic image of $\mathcal{S}/\equiv_{\mathrm{U}}$. We conclude that $\mathcal{L} \in \mathrm{HSP_u}(\mathbf{K})$. □

The proof of Jónsson's Lemma [12] uses the distributivity of **Con** $\mathcal{L}$ in a crucial way, and its conclusion is not generally true for varieties of algebras that do not have distributive congruence lattices. This means that varieties of lattices are more well-behaved than varieties of other algebras, such as groups and rings. The applications below will indicate some aspects of this.

Lemma 7.8. *Let U be an ultrafilter on $\mathfrak{P}(I)$ and $J \in U$. Then $V = \{B \subseteq J : B \in U\}$ is an ultrafilter on $\mathfrak{P}(J)$, and $\prod_{j \in J} \mathcal{L}_j/\equiv_{\mathrm{V}}$ is isomorphic to $\prod_{i \in I} \mathcal{L}_i/\equiv_{\mathrm{U}}$.*

Proof. V is clearly a proper filter. Moreover, if $A \subseteq J$ and $A \notin V$, then $I - A \in U$ and hence $J - A = J \cap (I - A) \in U$. It follows by Lemma 7.5(2) that V is an ultrafilter.

The projection $\rho_J : \prod_{i \in I} \mathcal{L}_i \twoheadrightarrow \prod_{j \in J} \mathcal{L}_j$ is a surjective homomorphism. As $A \cap J \in U$ if and only if $A \in U$, it induces a (well defined) isomorphism of $\prod_{i \in I} \mathcal{L}_i/\equiv_{\mathrm{U}}$ onto $\prod_{j \in J} \mathcal{L}_j/\equiv_{\mathrm{V}}$. □

Theorem 7.9. *Let $\mathbf{K} = \{\mathcal{K}_1, \ldots, \mathcal{K}_n\}$ be a finite collection of finite lattices. If $\mathcal{L}$ is a subdirectly irreducible lattice in the variety $\mathrm{HSP}(\mathbf{K})$, then $\mathcal{L} \in \mathrm{HS}(\mathcal{K}_j)$ for some j.*

Proof. By Jónsson's Lemma, $\mathcal{L}$ is a homomorphic image of a sublattice of an ultraproduct $\prod_{i \in I} \mathcal{L}_i/\equiv_{\mathrm{U}}$ with each $\mathcal{L}_i$ isomorphic to one of $\mathcal{K}_1, \ldots, \mathcal{K}_n$. Let $A_j = \{i \in I : \mathcal{L}_i \cong \mathcal{K}_j\}$. As $A_1 \cup \cdots \cup A_n = I \in U$, by Lemma 7.5(4) there is a j such that $A_j \in U$. But then Lemma 7.8 says that there is an ultrafilter V on $\mathfrak{P}(A_j)$ such that the original ultraproduct is isomorphic to $\prod_{k \in A_j} \mathcal{L}_k / \equiv_{\mathrm{V}}$, wherein each $\mathcal{L}_k \cong K_j$. However, for any finite lattice $\mathcal{K}$ there is a first order sentence $\varphi_{\mathcal{K}}$ such that a lattice $\mathcal{M}$ satisfies $\varphi_{\mathcal{K}}$ if and only if $\mathcal{M} \cong \mathcal{K}$. Therefore, by Łos' Theorem, $\prod_{k \in A_j} \mathcal{L}_k / \equiv_{\mathrm{V}}$ is isomorphic to $\mathcal{K}_j$. Hence $\mathcal{L} \in HS(\mathcal{K}_j)$, as claimed. □

Since a variety is determined by its subdirectly irreducible members, we have the following consequence.

Corollary. *If* $\mathbf{V} = \mathrm{HSP}(\mathbf{K})$ *where* $\mathbf{K}$ *is a finite collection of finite lattices, then* $\mathbf{V}$ *contains only finitely many subvarieties.*

Note that $\mathrm{HSP}(\{\mathcal{K}_1, \ldots, \mathcal{K}_n\}) = \mathrm{HSP}(\mathcal{K}_1 \times \cdots \times \mathcal{K}_n)$, so w.l.o.g. we can talk about the variety generated by a single finite lattice. The author has shown that the converse of the Corollary is false [18]: *There is an infinite, subdirectly irreducible lattice* $\mathcal{L}$ *such that* $\mathrm{HSP}(\mathcal{L})$ *has only finitely many subvarieties, each of which is generated by a finite lattice.*

There are many other consequences of Jónsson's Lemma, especially for varieties of modular lattices. Many contributors combined to develop an elegant theory of lattice varieties, which we will not attempt to survey. The standard reference on the subject is the book of Peter Jipsen and Henry Rose [11].

Let us call a variety $\mathbf{V}$ *finitely based* if $\mathbf{V} = V(\Sigma)$ for some finite set of equations Σ. These are just the varieties that are dually compact in the lattice Λ of lattice varieties. Ralph McKenzie [15] proved the following nice result.

Theorem 7.10. *The variety generated by a finite lattice is finitely based.*

Kirby Baker [1] generalized this result by showing that if $\mathcal{A}$ is any finite algebra in a variety $\mathbf{V}$ such that (i) $\mathbf{V}$ has only finitely many operation symbols, and (ii) the congruence lattices of algebras in $\mathbf{V}$ are distributive, then $\mathrm{HSP}(\mathcal{A})$ is finitely based. It is also true that the variety generated by a finite group is finitely based (S. Oates and M. B. Powell [19]), and likewise the variety generated by a finite ring (R. Kruse [13]). See R. McKenzie [16] for a common generalization of these finite basis theorems. There are many natural examples of finite algebras that do not generate a finitely based variety; see, e.g., G. McNulty [17]. A good survey of finite basis results is R. Willard [20].

We will return to the varieties generated by some particular finite lattices in the next chapter.

If $\mathbf{V}$ is a lattice variety, let $\mathbf{V}_{si}$ be the class of subdirectly irreducible lattices in $\mathbf{V}$. The next result is proved by a straightforward modification of the first part of the proof of Theorem 7.9.

Theorem 7.11. *If* $\mathbf{V}$ *and* $\mathbf{W}$ *are lattice varieties, then* $(\mathbf{V} \vee \mathbf{W})_{si} = \mathbf{V}_{si} \cup \mathbf{W}_{si}$.

Corollary. Λ *is distributive.*

Theorem 7.11 does not extend to infinite joins. For example, finite lattices generate the variety of all lattices (see Exercise 6) but there are infinite subdirectly irreducible lattices. We already knew the Corollary by Theorem 7.3, because Λ is dually isomorphic to a sublattice of **Con** FL(ω), which is distributive, but this provides an interesting way of looking at it.

In closing let us consider the lattice $\mathcal{I}(\mathcal{L})$ of ideals of $\mathcal{L}$. An elementary argument shows that the map $x \to \downarrow x$ embeds $\mathcal{L}$ into $\mathcal{I}(\mathcal{L})$. A classic theorem of Garrett Birkhoff [4] says that $\mathcal{I}(\mathcal{L})$ satisfies every identity satisfied by $\mathcal{L}$, i.e., $\mathcal{I}(\mathcal{L}) \in \mathrm{HSP}(\mathcal{L})$. The following result of Kirby Baker and Alfred Hales [2] goes one better.

Theorem 7.12. *For any lattice* $\mathcal{L}$*, we have* $\mathcal{I}(\mathcal{L}) \in \mathrm{HSP}_\mathrm{u}(\mathcal{L})$.

This is an ideal place to stop.

Exercises for Chapter 7

1. Show that fully invariant congruences form a complete sublattice of **Con** $\mathcal{L}$.

2. Let $\mathcal{L}$ be a lattice and **V** a lattice variety. Show that there is a unique minimum congruence $\rho_\mathbf{V}$ on $\mathcal{L}$ such that $\mathcal{L}/\rho_\mathbf{V} \in \mathbf{V}$.

3. (a) Prove that if $\mathcal{L}$ is a subdirectly irreducible lattice, then HSP($\mathcal{L}$) is (finitely) join irreducible in the lattice Λ of lattice varieties.

(b) Prove that if a variety **V** is completely join irreducible in Λ, then $\mathbf{V} = \mathrm{HSP}(\mathcal{K})$ for some finitely generated, subdirectly irreducible lattice $\mathcal{K}$.

4. Show that if F is a filter on $\mathfrak{P}(I)$, then $\equiv_F$ is a congruence relation on $\prod_{i \in I} \mathcal{L}_i$.

5. (a) Show that if F is a filter on $\mathfrak{P}(I)$, then F is the intersection of the ultrafilters U such that $U \supseteq F$.

(b) Then prove that the congruence $\equiv_F$ on $\prod_{i \in I} \mathcal{L}_i$ is the intersection of congruences $\equiv_U$ with U an ultrafilter.

(c) Conclude that the reduced product $\prod_{i \in I} \mathcal{L}_i / \equiv_F$ is a subdirect product of ultraproducts $\prod_{i \in I} \mathcal{L}_i / \equiv_U$.

6. Prove that every lattice equation that does not hold in all lattices fails in some finite lattice. (Let $p \neq q$ in FL(X). Then there exist a finite join subsemilattice $\mathcal{S}$ of FL(X) containing p, q and $0 = \bigwedge X$, and a lattice homomorphism $h : \mathrm{FL}(X) \to \mathcal{S}$, such that $h(p) = p$ and $h(q) = q$.)

The standard solution to Exercise 6 involves lattices which turn out to be lower bounded (see Exercise 11 of Chapter 6). Hence they satisfy $\mathrm{SD}_\vee$, and any finite collection of them generates a variety not containing $\mathcal{M}_3$, while all together they generate the variety of all lattices. On the other hand, the variety generated by $\mathcal{M}_3$ contains only the variety **D** of distributive lattices (generated by **2**) and the trivial variety **T**. It follows that the lattice Λ of lattice varieties is not join continuous.

7. Give a first order sentence characterizing each of the following properties of a lattice $\mathcal{L}$ (i.e., $\mathcal{L}$ has the property iff $\mathcal{L} \models \varphi$).

(a) $\mathcal{L}$ has a least element.
(b) $\mathcal{L}$ is atomic.
(c) $\mathcal{L}$ is strongly atomic.

(d) $\mathcal{L}$ is weakly atomic.
(e) $\mathcal{L}$ has no covering relations.

8. A lattice $\mathcal{L}$ has *breadth* n if L contains n elements whose join is irredundant, but every join of $n+1$ elements of L is redundant.

(a) Give a first order sentence characterizing lattices of breadth n (for a fixed finite integer $n \geq 1$).
(b) Show that the class of lattices of breadth $\leq n$ is not a variety.
(c) Show that a lattice $\mathcal{L}$ and its dual $\mathcal{L}^d$ have the same breadth.

9. Give a first order sentence φ such that a lattice $\mathcal{L}$ satisfies φ if and only if $\mathcal{L}$ is isomorphic to the four element lattice $\mathbf{2} \times \mathbf{2}$.

10. Prove Theorem 7.11.

11. Prove that $\mathcal{I}(\mathcal{L})$ is distributive if and only if $\mathcal{L}$ is distributive. Similarly, show that $\mathcal{I}(\mathcal{L})$ is modular if and only if $\mathcal{L}$ is modular.

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