Math 241
Fall 2018
Exam 1 - Practice
9/26/18
Time Limit: 50 Minutes

Name (Print): Dat Bat

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	15	
8	20	
9	15	
Total:	170	

Homework:
Recitation:
Current Class Grade:

1. (20 points) Find the following limits:

a) $\lim_{x\to 2} \frac{(x^2-4)(x+2)}{(x^2-5x+6)(x+1)}$ $= \lim_{x\to 2} \frac{(x+2)(x-2)(x+2)}{(x-3)(x-2)(x+1)}$

$= \lim_{x\to 2} \frac{(x+2)(x+2)}{(x-3)(x+1)}$

$= \frac{16}{-1\cdot 3} = \boxed{-\frac{16}{3}}$

b) $\lim_{x\to 0} \frac{\tan(2x)}{3x}$ $= \lim_{x\to 0} \frac{\sin(2x)}{\cos(2x)} \cdot \frac{1}{3x} \cdot \frac{2}{2}$

$= \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \boxed{\frac{2}{3}}$

c) $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$ $= \lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1} \frac{(\sqrt{x+8}+3)}{(\sqrt{x+8}+3)} = \frac{x+8-9}{(x-1)(\sqrt{x+8}+3)}$

$= \frac{1}{\sqrt{x+8}+3}$

$= \frac{1}{6}$

d) $\lim_{x\to 1^-} \frac{\sqrt{3x}|x-1|}{(x-1)}$ if $x \to 1^-$ then $x<1 \Rightarrow 0 > x-1$

$\Rightarrow |x-1| = -(x-1)$

$= \lim_{x\to 1^-} \frac{\sqrt{3x}(-(x-1))}{(x-1)}$

$= \lim_{x\to 1^-} -\sqrt{3x} = -\sqrt{3}$

2. (20 points) Find the following limits:

a) $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}+\sqrt{x^2+x}} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2(1+\frac{1}{x^2})} + \sqrt{x^2(1+\frac{1}{x})}}$

$= \lim_{x\to-\infty} \frac{x}{|x|\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x}}\right)}$

$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x}}} = -\frac{1}{2}$

This step uses the fact that $|x| = -x$ when $x<0$.

b) $\lim_{x\to\infty} \frac{2\sin(x)}{x^2+1}$

Since

$-1 \le \sin(x) \le 1$

we have $-2 \le 2\sin(x) \le 2$

and therefore $\frac{-2}{x^2+1} \le \frac{2\sin(x)}{x^2+1} \le \frac{2}{x^2+1}$

Now, because $\lim_{x\to\infty} \frac{-2}{x^2+1}$

$= \lim_{x\to\infty} \frac{-2/x^2}{1+\frac{1}{x^2}} = 0$

and $\lim_{x\to\infty} \frac{2}{x^2+1} = 0$ for the same reason,

The squeeze theorem gives us

$\lim_{x\to\infty} \frac{2\sin(x)}{x^2+1}$.

c) $\lim_{x\to2^-} \frac{x^2-10}{x^2-4}$

This is a rational function w/ type $\frac{\#\neq 0}{0}$,

<u>and</u> a one sided limit, the answer is therefore $+\infty$ or $-\infty$.

As $x^2-10<0$ at $x=2$ and $x^2-4<0$ as $x\to2^-$,

$\lim_{x\to2^-} \frac{x^2-10}{x^2-4} = +\infty$

d) $\lim_{x\to\infty} \frac{(x^{5/3}-10)\sqrt[3]{x}}{2x^2+x-4}$

$= \lim_{x\to\infty} \frac{x^2-10\sqrt[3]{x}}{2x^2+x-4} = \lim_{x\to\infty} \frac{x^2\left(1-\frac{10}{x^{5/3}}\right)}{x^2\left(2+\frac{1}{x}-\frac{4}{x^2}\right)}$

$= \frac{1}{2}$

3. (20 points) a) State the Intermediate Value Theorem.

Suppose that f is continuous on $[a,b]$ and c is a number inbetween $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number, α, in the interval (a,b) such that $f(\alpha) = c$.

b) Use the Intermediate Value Theorem to show that the equation $\sin(x)+x = 1$ has a solution.

Let $f(x) = \sin(x) + x - 1$, and note that f is continuous. Since $f(0) = -1$ and $f(\pi) = \pi - 1$ (which is positive), the number 0 is inbetween -1 and $\pi - 1$. The I.V.T. gives us some number α such that $f(\alpha) = 0$.

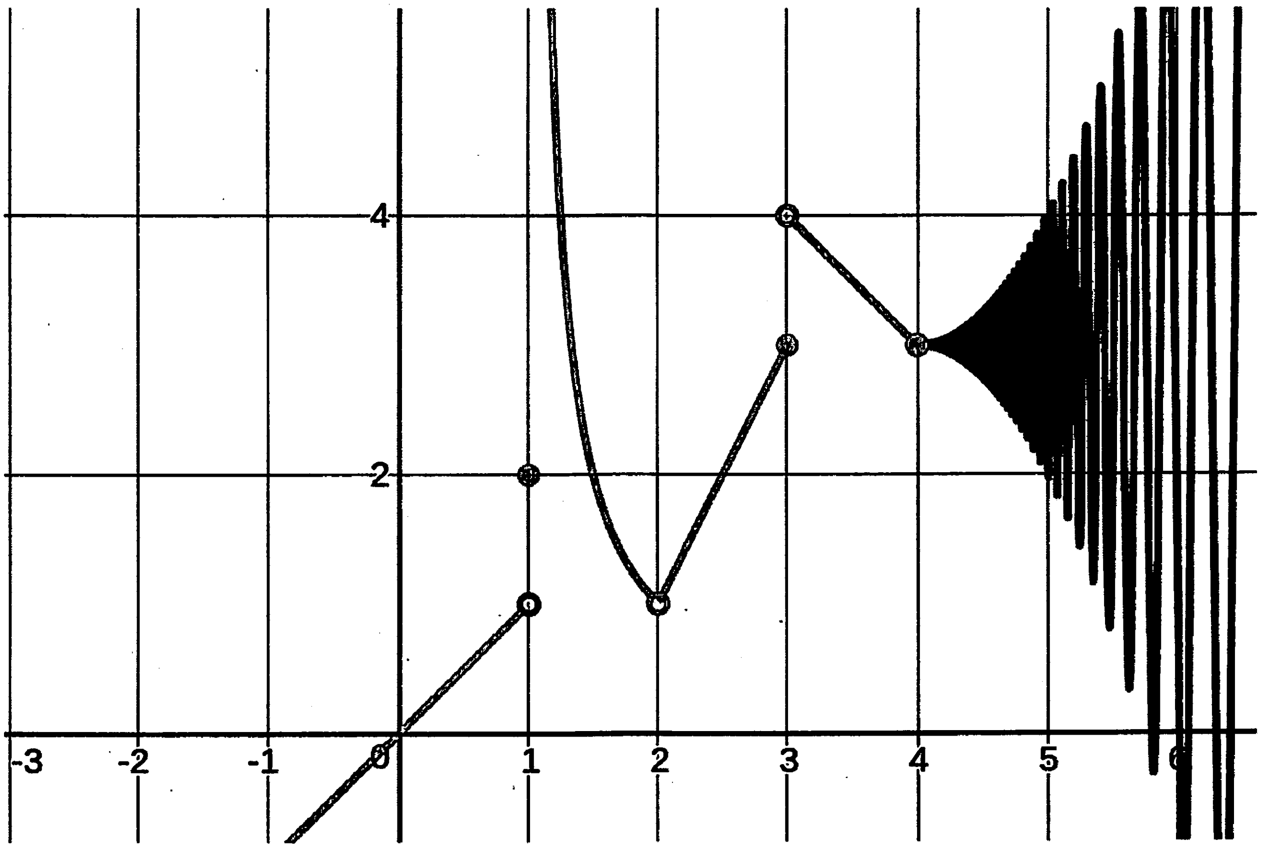
We have

$$0 = f(\alpha) = \sin(\alpha) + \alpha - 1$$

$$\Rightarrow \quad 1 = \sin(\alpha) + \alpha$$

and therefore α is a solution to the equation.

4. (20 points) Below is a graph of $y = f(x)$. For $0 \leq x \leq 5$, state where the following function fails to be continuous and why.



$f(x)$ is not continuous at

$x = 1$ because $\lim_{x \to 1} f(x)$ does not exist.

$x = 2$ $f(2)$ is not defined (note: This is called a removable discontinuity)

$x = 3$ $\lim_{x \to 3} f(x)$ does not exist. (note: This is called a jump discontinuity)

5. (20 points) Use the **definition of the derivative as a limit** to find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt{x+1}}$.
Note: using derivative rules will get no points!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{\sqrt{x+h+1}} - \dfrac{1}{\sqrt{x+1}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{\sqrt{x+h+1} \cdot \sqrt{x+1} \cdot h}$$

$$= \lim_{h \to 0} \frac{x+1 - (x+h+1)}{\sqrt{x+h+1} \cdot \sqrt{x+1} \cdot h\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \frac{-1}{\left(\sqrt{x+1}\right)^2 \cdot 2\left(\sqrt{x+1}\right)}$$

$$= \frac{-1}{2\left(\sqrt{x+1}\right)^3}$$

6. Find the following derivatives: (warning! no partial credit)

(a) (5 points) $f(x) = \sqrt{x} + 7x + \frac{1}{x}$

$$f'(x) = \frac{1}{2\sqrt{x}} + 7 - \frac{1}{x^2}$$

(b) (5 points) $g(x) = \tan(x)\sqrt{x^2+1}$

$$g'(x) = \sec^2(x)\sqrt{x^2+1} + \frac{1}{2\sqrt{x^2+1}} \cdot 2x \cdot \tan(x)$$

(c) (5 points) $h(x) = \dfrac{3\sec(2x)}{\frac{1}{x^2}+x}$

$$h'(x) = \frac{3\sec(2x)\tan(2x) \cdot 2\left(\frac{1}{x^2}+x\right) - \left(\frac{-2}{x^3}+1\right) \cdot 3\sec(2x)}{\left(\frac{1}{x^2}+x\right)^2}$$

(d) (5 points) $k(x) = \sin(1+\sqrt{x^2+2})$

$$k'(x) = \cos\left(1+\sqrt{x^2+2}\right) \cdot \frac{1}{2\sqrt{x^2+2}} \cdot 2x$$

7. (15 points) Given $x^2 + y^3 = xy^2 + 1$, find $\frac{dy}{dx}$ and give the equation of both the normal and tangent lines at the point $(1, 1)$.

$$\frac{d}{dx}\left(x^2 + y^3\right) = \frac{d}{dx}\left(xy^2 + 1\right)$$

$$2x + 3y^2 \cdot \frac{dy}{dx} = y^2 + x\,2y\,\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(3y^2 - x2y\right) = y^2 - 2x$$

$$\frac{dy}{dx} = \frac{y^2 - 2x}{3y^2 - 2xy}, \quad \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-1}{1} = -1$$

Tangent line at $(1,1)$:

$$y - 1 = -1(x - 1)$$

Normal line at $(1,1)$:

$$y - 1 = 1(x - 1)$$

8. (a) (10 points) Given the position function $p(t) = \sin(t) + \frac{1}{2}t$ (in feet where t is in seconds), find the velocity function $v(t)$, and the acceleration function $a(t)$. On the interval $[0, 2\pi]$, when is the object's velocity positive?

$$v(t) = p'(t) = \cos(t) + \frac{1}{2} \qquad a(t) = v'(t) = -\sin(t)$$

$v(t) > 0 \Rightarrow \cos(t) > -\frac{1}{2} \Rightarrow t$ is in the interval $[0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi)$

(b) (10 points) Given the position function $p(t) = \dfrac{1-t^2}{t^2+4}$ (in feet where t is in seconds). Determine when the object is

a) At rest

$$v(t) = p'(t) = \frac{-2t(t^2+4) - (1-t^2)2t}{(t^2+4)^2} = \frac{2t(-t^2-4-1+t^2)}{(t^2+4)^2} = \frac{-10t}{(t^2+4)^2}$$

so, $v(t) = 0$ when $t = 0$.

b) Moving forward

moving forward → $v(t) > 0$ when $t < 0$, $(-\infty, 0)$

$v(t) < 0$ when $t > 0$, $(0, \infty)$

c) Moving backward

moving backward

d) When is the acceleration positive? When is it negative?

$$a(t) = \frac{-10(t^2+4)^2 + 10t(2(t^2+4)\cdot 2t)}{(t^2+4)^2}$$

$$= \frac{(t^2+4)[-10(t^2+4) + 40t^2]}{(t^2+4)^2}$$

$$= \frac{30t^2 - 40}{(t^2+4)^2} = \frac{30(t^2 - \frac{4}{3})}{(t^2+4)^2}$$

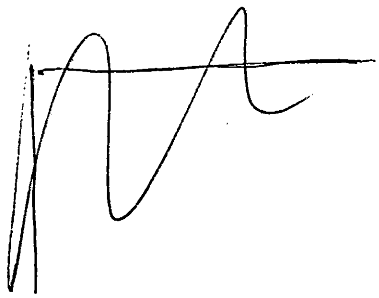
$a(t) > 0$ on $(-\infty, \frac{-2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$, and $a(t) < 0$ on $(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

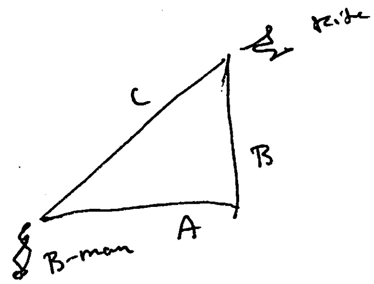
e) When is the object speeding up?

$$(-\infty, \frac{-2}{\sqrt{3}}) \cup (0, \frac{2}{\sqrt{3}})$$

9. (15 points) Batman is particular about kites. He ONLY flies a kite at a height of exactly $300ft.$, always. Today is no different. Also, today, the wind pushes the kite and it moves horizontally in the air at a rate $25\frac{ft}{sec.}$.

a) How fast must he be letting out string for the kite to remain at the constant height of $300ft.$ at the precise moment he has let $500\ ft.$ of string out?





$$\frac{dA}{dt} = 25$$

$C = 500$

$B = 300$

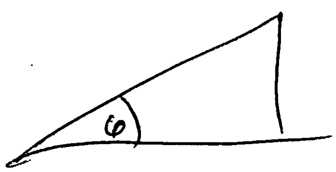
$A = 400$

$$c^2 = A^2 + B^2$$

$$2c\frac{dc}{dt} = 2A\frac{dA}{dt} + 0$$ (B is constant)

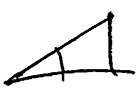
$$\frac{dc}{dt} = \frac{400\cdot 25}{500} = 20 \text{ ft/sec.}$$

b) How fast is the angle between the string and the ground changing at this time?



$$\tan(\theta) = \frac{B}{A} = 300\cdot A^{-1}$$

$$\sec^2(\theta)\cdot\frac{d\theta}{dt} = -300A^{-2}\cdot\frac{dA}{dt}$$



$$\cos(\theta) = \frac{400}{500} = \frac{4}{5}$$

$$\frac{d\theta}{dt} = -300\cdot\left(\frac{1}{(400)^2}\right)\cdot 25\,\frac{1}{\sec^2(\theta)}$$

$$= -\frac{3}{4}\,\frac{1}{400}\cdot 25\left(\frac{4}{5}\right)^2 = \frac{-3}{100}\ \text{rad/sec.}$$