Math 241
Fall 2018
Exam 2 - Practice
10/26/18
Time Limit: 50 Minutes

Name (Print): Solutions
Section:

Problem	Points	Score
1	20	
2	30	
3	30	
4	15	
5	20	
6	10	
7	30	
8	40	
Total:	195	

1. (20 points) Use linearization to approximate $\sqrt{143}$ and $\sqrt[3]{124}$.

$\sqrt{143}$: Let $f(x) = \sqrt{x}$. We aim to linearize about $x = 144$.

$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(144) = \frac{1}{2\sqrt{144}} = \frac{1}{24}$, $f(144) = \sqrt{144} = 12$

$L(x) = \frac{1}{24}(x - 144) + 12$ is thus the equation of the tangent line of $f(x)$ @ $x = 144$.

$$\sqrt{143} \approx L(143) = \frac{1}{24}(143 - 144) = \frac{-1}{24} + 12$$

$\sqrt[3]{124}$: $f(x) = \sqrt[3]{x}$, $f'(x) = \frac{1}{3x^{2/3}}$

$f(125) = 5$, $f'(125) = \frac{1}{3(\sqrt[3]{125})^2} = \frac{1}{75}$

$L(x) = \frac{1}{75}(x - 125) + 5$

$$\sqrt[3]{124} \approx L(124) = \frac{-1}{75} + 5$$

2. (a) (10 points) State Rolle's Theorem and the Mean Value Theorem.

See notes/textbook.

(b) (10 points) Use the IVT and Rolle's Theorem (or the Mean Value Theorem) to show that $2x - \sqrt{2} = \cos^2(x)$ has one, and only one solution.

Let $f(x) = 2x - \cos^2(x) - \sqrt{2}$. Since $f(0) = -1 - \sqrt{2} < 0$ and $f(\frac{3\pi}{2}) = 3\pi - \sqrt{2} > 0$, the IVT gives us an x_0 in $[0, \frac{3\pi}{2}]$ such that $f(x_0) = 0$.

If there is ~~an~~ more than one solution to the given equation, our function f would have more than one zero, say, at $x = x_1$. In this case Rolle's Theorem gives us a c in (x_0, x_1) such that $f'(c) = 0$. However, $f'(x) = 2 + 2\cos(x)\sin(x) = 2 + \sin(2x) > 0$, which is a contradiction. Oh yeah, f is continuous and differentiable, so, we are indeed allowed to use IVT/MVT. All this gives us that the equation only has one solution.

(c) (10 points) Show that if f is differentiable at $x = c$, then f is continuous at $x = c$.

We show that $\lim_{x \to a} f(x) = f(a)$:

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

and this gives

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a)$$

$$= \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right) \cdot \lim_{x \to a}(x - a)$$

$$= f'(a) \cdot 0$$

$$= 0$$

whence, $\lim_{x \to a} f(x) = f(a)$!

3. (30 points) True of false: If $f''(c) = 0$ for some c, then f has an inflection point at $(c, f(c))$.

For a counter-example, try $f(x) = x^4$.

True of false: If f is differentiable on (a, b) and continuous of $[a, b]$ then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for any c in (a, b).

Counter example: Any function that is not a line.

True of false: If $f'(x) = 0$ then f is a constant function.

This is a consequence of the MVT.

True of false: If $f'(c) = 0$ for some c, then $x = c$ is a critical number for f at $x = c$.

We would only be worried about f being defined at $x = c$, but as $f'(c) = 0$, f is differentiable at $x = c$ and thefore continuous and whence defined.

True of false: If $f'(c) = 0$ for some c, then f has a local min or max at $x = c$.

Counter Example: $f(x) = x^3$.

4. (15 points) a) Find the absolute extrema of $f(x) = x^{2/3}(x-6)$ on the interval $[-1, 5]$.

$f'(x) = \dfrac{5x - 12}{3x^{1/3}}$ (why?) C.P.s @ $x = 0$ and $x = 12/5$

$f(-1) = -7$ (abs min)

$f(0) = 0$ (abs max)

$f(12/5) \approx -6.45$

$f(5) = -\sqrt[3]{25}$

Note: $f(12/5)$ is a bit too tricky for the actual exam.

b) Find the absolute extrema of $f(x) = (x-3)^{2/3}$ on the interval $[2, 11]$.

$f(x) = \dfrac{2}{3(x-3)^{1/3}}$ C.P. @ $x = 3$.

$f(2) = 1$

$f(3) = 0 \longleftarrow$ abs. min.

$f(11) = (\sqrt[3]{8})^2 = 4 \longleftarrow$ abs. max.

c) Find the absolute extrema of $f(x) = \frac{x^3}{3} - 2x^2 + 3x$ on the interval $[0, 4]$.

$f'(x) = x^2 - 4x + 3 = (x-1)(x-3)$

$f(0) = 0$ (abs min)

$f(1) = 4/3$ (abs max)

$f(3) = 0$ (abs min)

$f(4) = \frac{4}{3}$ (abs max)

5. (20 points) Consider the function $f(x) = \dfrac{x^2-1}{x^2+1}$.

a) Determine the interval(s) where $f(x)$ is positive/negative.

$f(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$

and $f(x) < 0$ on $(-1, 1)$.

b) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Give the equations of any asymptotes (horizontal, vertical or slant).

$$\lim_{x\to\pm\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\pm\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = 1$$

$\therefore$ $y=1$ is the equation of a horizontal asymptote.

c) Give the intervals of increase and decrease and give the coordinates of any local min/max (meaning the x and y values).

$$f'(x) = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

f': $-$ on the left of 0, $+$ on the right of 0

So, $f(x)$ is increasing on $(0,\infty)$ and decreasing on $(-\infty, 0)$. The first derivative test ~~gives~~ tells us that f has a local min at $x=0$.

d) Find the intervals of concavity and the coordinates of any inflection points.

$$f''(x) = \frac{4(x^2+1)^2 - 2(x^2+1)\cdot 2x\cdot 4x}{(x^2+1)^4} = \frac{4(x^2+1) - 16x^2}{(x^2+1)^3}$$

$$= \frac{4(1-3x^2)}{(x^2+1)^3}$$

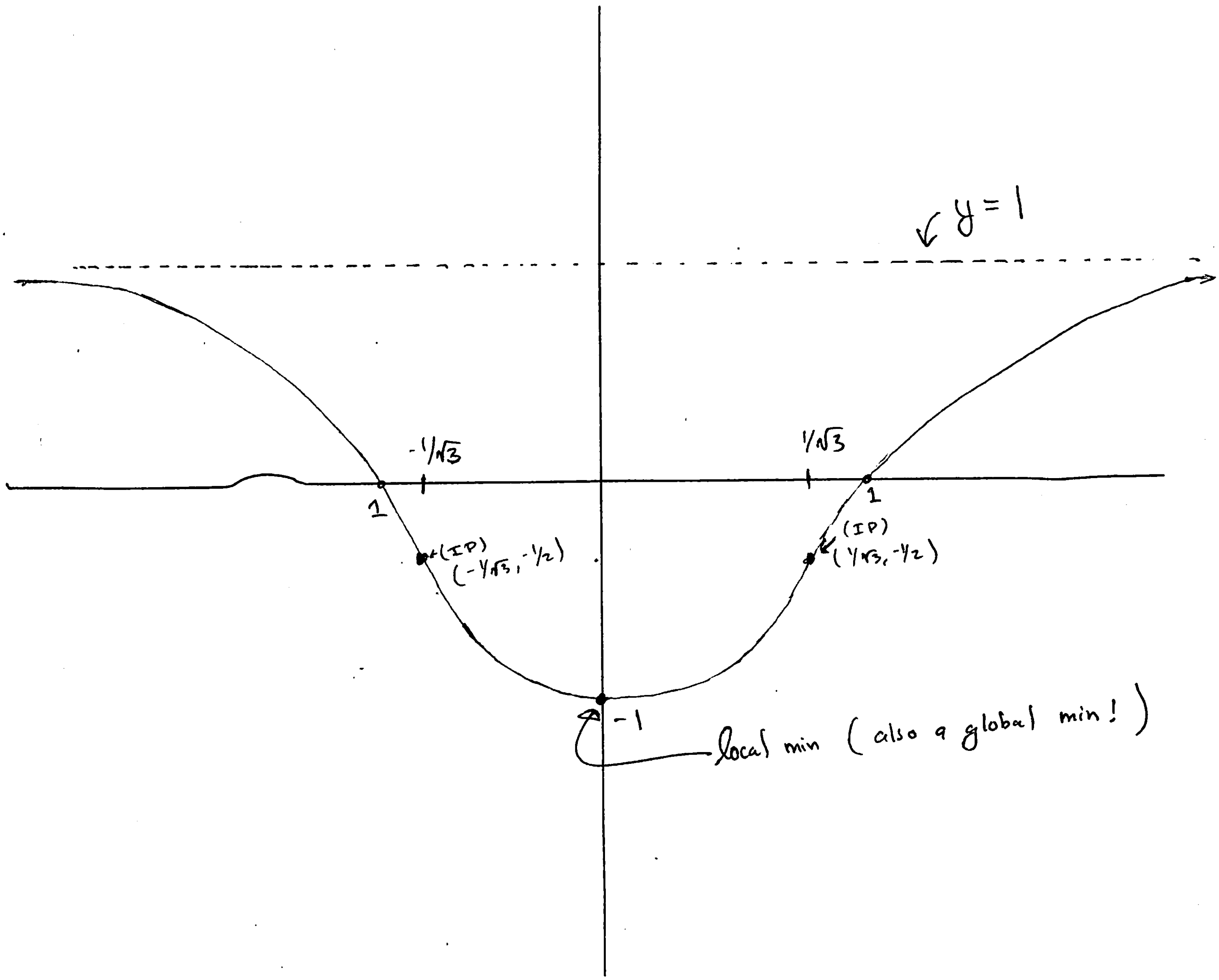
f'': $-$ on the left of $\frac{-1}{\sqrt{3}}$, $+$ between $\frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$, $-$ on the right of $\frac{1}{\sqrt{3}}$

f is C.U. on $(-1/\sqrt{3}, 1/\sqrt{3})$ and

C.D. on $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$.

There are two I.P.s @ $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{2}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{-1}{2}\right)$.

6. (10 points) Sketch a graph of the function from the previous page. Label the asymptotes, extrema and inflection points.



7. (30 points) Repeat the process of problems 5 and 6 for the functions $f(x) = \dfrac{x^2}{\sqrt{x+1}}$, $g(x) = \dfrac{2x^2}{x^2-1}$ and $h(x) = \dfrac{x^2+3}{x-1}$.

f(x) and g(x) are both in your book in section 3.5.

h(x): $h'(x) = \dfrac{2x(x-1) - (x^2+3)}{(x-1)^2} = \dfrac{x^2-2x-3}{(x-1)^2} = \dfrac{(x+1)(x-3)}{(x-1)^2}$

Clearly, there is a V.A. @ $x=1$.

So, (note that the domain of h is $(-\infty,1)\cup(1,\infty)$)

h': $+ \quad - \quad - \quad +$ at $-1 \quad 1 \quad 3$

$h(x) < 0$ on $(-\infty,1)$ and $h(x) > 0$ on $(1,\infty)$.

h is increasing on $(-\infty,-1)\cup(3,\infty)$ and decreasing on $(-1,1)\cup(1,3)$.

Notice that

$$\begin{array}{r|l} & x+1 \\ x-1 & x^2+3 \\ & -(x^2-x) \\ \hline & 0\; x+3 \\ & -(x-1) \\ \hline & 4 \end{array}$$

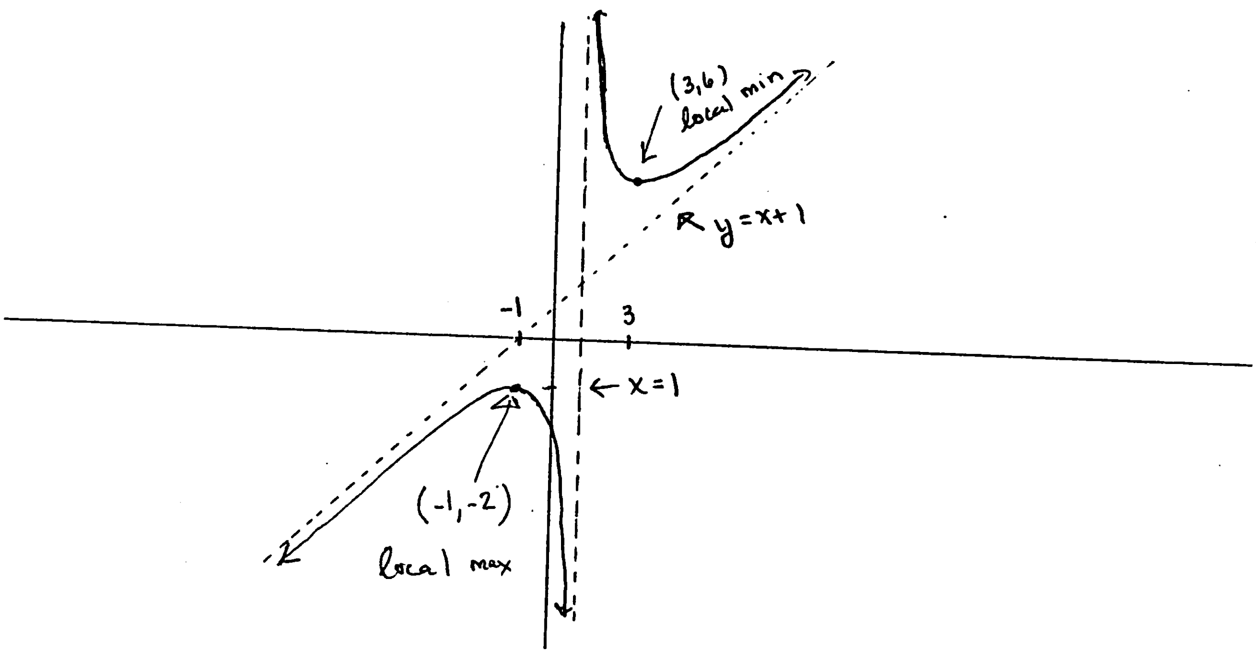
and therefore $h(x) = x+1+\dfrac{4}{x-1}$ and so has a slant asymptote: $y = x+1$.

$$h''(x) = \frac{(2x-2)(x-1)^2 - 2(x-1)(x^2-2x-3)}{(x-1)^4} = \frac{2(x-1)\left[(x-1)^2 - (x^2-2x-3)\right]}{(x-1)^4}$$

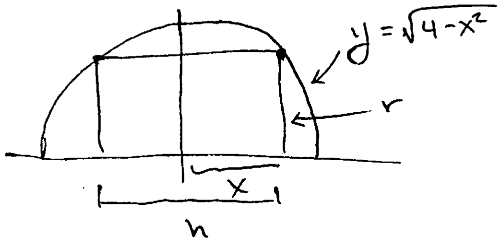
$$= \frac{8}{(x-1)^3}$$

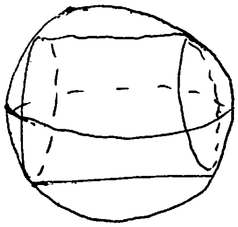
h'': $- \quad +$ at 1

So, there is not an I.P. at $x=1$ (not in domain of f(x))!



8. (40 points) a) Find the radius and height of the largest right circular cylinder that can fit inside a sphere of radius 2.





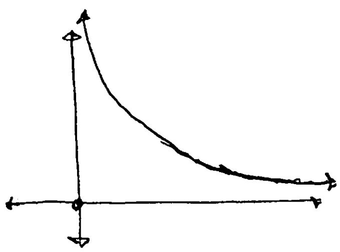
$$V = \pi r^2 h$$
$$= \pi \left(\sqrt{4-x^2}\right)^2 \cdot 2x \qquad (\mathcal{D}: [0,2])$$
$$= \pi\left(8x - 2x^3\right)$$

$$V'(x) = \pi\left(8 - 6x^2\right) \therefore V' \quad + \;\; \sqrt{4/3} \;\; -$$

So, The largest is @ $x = \sqrt{4/3}$.

This gives $h = 2\sqrt{4/3} = \frac{4}{\sqrt{3}}$ and $r = \sqrt{4 - (\sqrt{4/3})^2} = \sqrt{8/3}$

b) What point on the graph of $f(x) = \frac{1}{\sqrt{x}}$ is closest to the origin?



$$d(x) = \sqrt{\left(\frac{1}{\sqrt{x}}\right)^2 - (x)^2}$$
$$= \sqrt{\frac{1}{x} + x^2}. \qquad \mathcal{D}: (0,\infty)$$

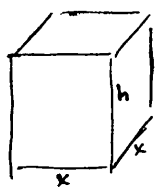
$$d'(x) = \frac{1}{2\sqrt{\frac{1}{x}+x^2}} \cdot \frac{-1}{x^2} + 2x$$

$$= \frac{2x^3 - 1}{2x^2\sqrt{\frac{1}{x}+x^2}} \qquad \therefore \text{ c.p. @ } x = \sqrt[3]{1/2}$$

$$d' \quad - \;\; \sqrt[3]{1/2} \;\; +$$

$\therefore$ Closest point is $\left(\frac{1}{\sqrt[3]{2}}, \sqrt[6]{2}\right)$

c) You are to design a, quite large, square bottom box with total volume of 1500 $ft.^3$. The mysterious material you are to use costs 2 dollars per $ft.^2$ and you need to use two sheets of mysterious material on the bottom (this makes the box stronger). Find the dimensions and cost of the cheapest box you can make.



Given $hx^2 = 1500 \Rightarrow h = \frac{1500}{x^2}$

Cost Function: $2(3x^2) + 2(4xh)$

$\Rightarrow C(x) = 6x^2 + 8x\left(\frac{1500}{x^2}\right)$

$= 6x^2 + \frac{12000}{x}$ $\mathcal{D}: (0, \infty)$

$C'(x) = 12x - \frac{12000}{x^2}$

$= \frac{12x^3 - 12000}{x^2} \Rightarrow x = 10$

C': $-$ | 10 | $+$ so, it's a min.

The dim's are $10 \times 10 \times 15$.