

Math 243
Spring 2019
Practice Exam 1
Doomsday
Time Limit: Probably Not Enough

Name (Print): Answer Key

| Problem | Points | Score |
|---------|--------|-------|
| 1 | 15 | |
| 2 | 30 | |
| 3 | 40 | |
| 4 | 15 | |
| 5 | 20 | |
| 6 | 20 | |
| 7 | 20 | |
| Total: | 160 | |

1. (15 points) Let $P = (1, 2, 3)$ and $Q = (0, -1, 2)$.

a) Find the distance between P and Q .

$$\overrightarrow{PQ} = \cancel{\text{Vector}} \langle 0-1, -1-2, 2-3 \rangle = \langle -1, -3, -1 \rangle$$

$$|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (-3)^2 + (-1)^2} = \boxed{\sqrt{11}}$$

b) Give the equation of a sphere, centered at P , that has the point Q on its surface.

$$\underline{\text{Center: }} (1, 2, 3), \quad \underline{\text{Radius: }} \sqrt{11}$$

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 11$$

c) Find the vector $\overrightarrow{PQ} = \langle -1, -3, -1 \rangle$

d) Find the angle between \vec{P} and \vec{Q} . $\theta = \cos^{-1} \left(\frac{\vec{P} \cdot \vec{Q}}{|\vec{P}| |\vec{Q}|} \right)$

$$\begin{aligned} \vec{P} \cdot \vec{Q} &= 1 \cdot 0 + 2 \cdot (-1) + 3 \cdot 2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} |\vec{P}| &= \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \\ |\vec{Q}| &= \sqrt{0^2 + (-1)^2 + (2)^2} = \sqrt{5} \end{aligned}$$

$$\begin{aligned} &= \cos^{-1} \left(\frac{4}{\sqrt{14} \sqrt{5}} \right) \end{aligned}$$

e) Parametrize (with parametric equations) the line segment which starts at P and ends at Q .

$$\begin{aligned} \vec{P}(1-t) + \vec{Q}t &= \langle (1-t) + 0t, 2(1-t) + (-1)t, 3(1-t) + 2t \rangle \\ &= \langle 1-t, 2-3t, 3-t \rangle \end{aligned}$$

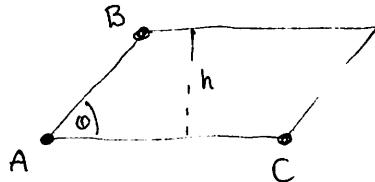
$$x = 1-t, \quad y = 2-3t, \quad z = 3-t \quad \text{for } 0 \leq t \leq 1.$$

Note: You can also use the vector \overrightarrow{PQ} for the "slope" and the point P as the "initial" position.

2. The following points define the vertices of a triangle:

$$A = (0, 0, 1) \quad B = (1, 3, -1) \quad C = (2, 2, 2)$$

(a) (10 points) Find the area of the triangle.



$$\vec{AB} = \langle 1, 3, -2 \rangle$$

$$\vec{AC} = \langle 2, 2, 1 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ 2 & 2 & 1 \end{vmatrix} = (3+4)i - (1+4)j + (2-6)k$$

$$= 7i - 5j - 4k$$

Area of the parallelogram is

$$\begin{aligned} |\vec{AC}| \cdot h &= |\vec{AC}| \cdot |\vec{AB}| \sin(\theta) \\ &= |\vec{AC} \times \vec{AB}| \\ &= |\vec{AB} \times \vec{AC}| \end{aligned}$$

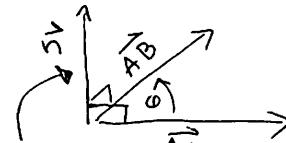
$$\text{Area of Parallelogram is: } \sqrt{7^2 + (-5)^2 + (-4)^2}$$

$$= \sqrt{90} = 3\sqrt{10}$$

so, the area of the triangle is

$$\frac{3\sqrt{10}}{2}$$

(b) (10 points) Find an equation of the plane which contains the triangle.



This vector is $\vec{AC} \times \vec{AB}$, since I calculated $\vec{AB} \times \vec{AC}$ in the previous problem,

$$\vec{n} = \vec{AC} \times \vec{AB} = -7i + 5j + 4k \quad (\text{and this is our normal vector})$$

$$\text{Equation: } -7(x-0) + 5(y-0) + 4(z-1) = 0.$$

Note: I could have used the vector $\vec{AB} \times \vec{AC}$, and any point.

(c) (10 points) Find the distance between the point $\underbrace{(3, 2, 1)}_{P}$ and the plane from part b).

$$\vec{AP} = \langle 3-0, 2-0, 1-1 \rangle = \langle 3, 2, 0 \rangle$$

$$\begin{aligned} d &= |\text{proj}_{\vec{n}} \vec{AP}| = \frac{|\vec{n} \cdot \vec{AP}|}{|\vec{n}|} = \frac{|3(-7) + 2(5) + 0 \cdot 4|}{3\sqrt{10}} \\ &= \frac{11}{3\sqrt{10}} \end{aligned}$$

3. (a) (10 points) Let $u = i + j$, $v = i + j + k$. Find the projection of u onto $u + v$.

$$u+v = 2i + 2j + k, \quad \text{so}$$

$$\begin{aligned}\text{Proj}_{u+v} u &= \frac{(u+v) \cdot u}{|u+v|^2} (u+v) \\ &= \frac{4}{9} (2i + 2j + k)\end{aligned}$$

- (b) (10 points) The vectors $u = i + 2j$ and $v = j + 3k$ lie in a plane that goes through the point $P = (1, 0, 1)$. Give the equation of this plane.

$$u \times v = 6i - 3j + k \quad (\text{This is our normal vector})$$

$$6(x-1) - 3(y-0) + 1(z-1) = 0$$

- (c) (10 points) Give the equation of a line, perpendicular to the plane $2x + 3y + z = 6$, that goes through the point $(1, 0, 1)$.

A ~~the~~ normal vector for the plane is $\langle 2, 3, 1 \rangle$, and we can use this vector as our "slope":

$$x = 1 + 2t, \quad y = 0 + 3t, \quad z = 1 + t$$

- (d) (10 points) Find the point in space the line from part c) intersects the plane from part b).

Why am I not going to answer this question?

4. (15 points) Match the surface with its equation

$$1 = x^2 + y^2 - z^2 \quad \text{f)}$$

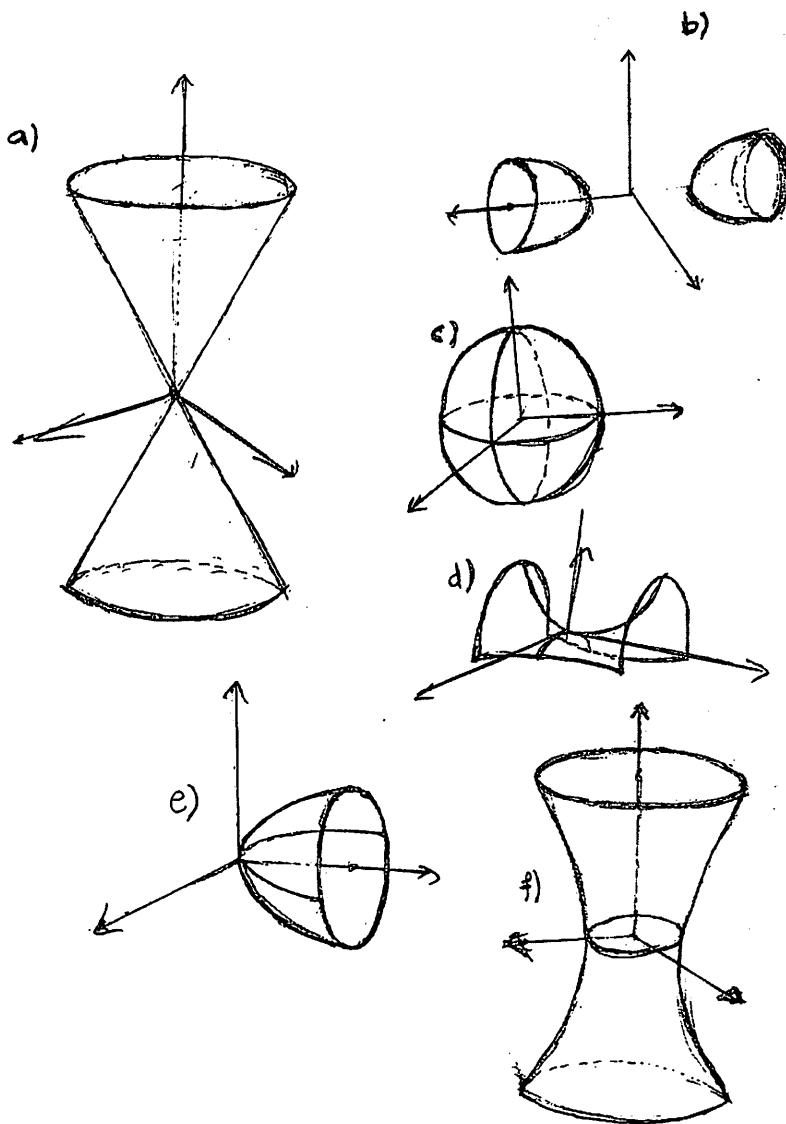
$$y = x^2 + z^2 \quad \text{e)}$$

$$1 = x^2 + y^2 + z^2 \quad \text{c)}$$

$$z = y^2 - x^2 \quad \text{d)}$$

$$0 = x^2 + y^2 - z^2 \quad \text{a)}$$

$$x^2 = z^2 + y^2 + 1 \quad \text{b)}$$



5. (20 points) Find the length of the curve given by the parametric equations

$$x = \cos(t) \quad \text{and} \quad y = t + \sin(t) \quad \text{for } 0 \leq t \leq \pi.$$

Also, find the equation of the tangent line when $t = \frac{\pi}{2}$.

The length of the curve is

$$\begin{aligned} & \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^\pi \sqrt{(-\sin(t))^2 + (1+\cos(t))^2} dt \\ &= \int_0^\pi \sqrt{\sin^2(t) + 1 + 2\cos(t) + \cos^2(t)} dt \\ &= \int_0^\pi \sqrt{2 + 2\cos(t)} dt \\ &= \int_0^\pi \sqrt{2 \cdot \left(2\cos^2\left(\frac{t}{2}\right)\right)} dt \\ &= 2 \int_0^\pi |\cos\left(\frac{t}{2}\right)| dt, \quad \text{but } \cos\left(\frac{t}{2}\right) > 0 \text{ on } [0, \pi], \\ & \quad \text{so we can drop the absolute value!} \\ &= 2 \int_0^\pi \cos\left(\frac{t}{2}\right) dt \\ &= 4 \sin\left(\frac{t}{2}\right) \Big|_{t=0}^{t=\pi} \\ &= 4 \end{aligned}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \cos(t)}{-\sin(t)}, \quad \text{and so } \frac{dy}{dx} \Big|_{t=\frac{\pi}{2}} = -1.$$

So, the equation of the tangent line when $t = \frac{\pi}{2}$ is

$$y - \left(\frac{\pi}{2} + 1\right) = -1(x - 0)$$

6. (20 points) Find the length of the curve

$$x = \frac{y^3}{6} + \frac{1}{2y} \rightarrow \text{let's parametrize this curve as}$$

from $y = 2$ to $y = 3$.

$$x = \frac{t^3}{6} + \frac{1}{2t}, \quad y = t$$

for $2 \leq t \leq 3$.

Now, we have $\frac{dx}{dt} = \frac{t^2}{2} + \frac{-1}{2t^2}$ and $\frac{dy}{dt} = 1$.

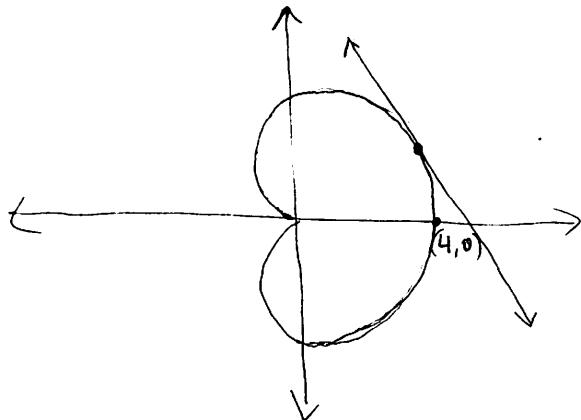
The length is thus

$$\begin{aligned} \int_2^3 \sqrt{\left(\frac{t^2}{2} - \frac{1}{2t^2}\right)^2 + 1^2} dt &= \int_2^3 \sqrt{\frac{t^4}{4} - \frac{1}{2} + \frac{1}{4t^4} + 1} dt \\ &= \int_2^3 \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2} dt \\ &= \int_2^3 \frac{t^2}{2} + \frac{1}{2t^2} dt \quad (\text{and maybe we say a small prayer for the absolute values here}) \\ &= \frac{t^3}{6} - \frac{1}{2t} \Big|_{t=2}^{t=3} \\ &= \frac{13}{4} \end{aligned}$$

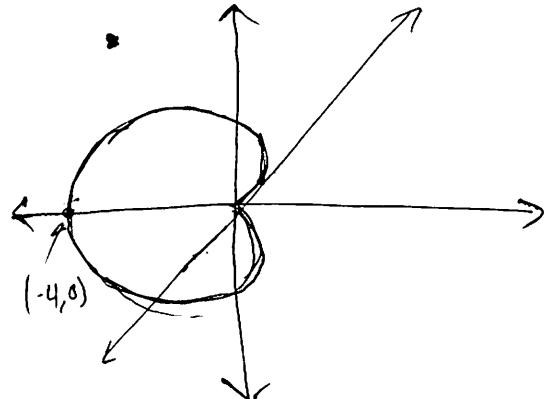
7. (20 points) Consider the polar coordinate equations $r = 2(1 + \cos(\theta))$ and $r = 2(1 - \cos(\theta))$.

a) Graph both of these curves. For both graphs, find and plot the equation of the tangent line when $\theta = \frac{\pi}{6}$.

$$r = 2(1 + \cos(\theta))$$



$$r = 2(1 - \cos(\theta))$$



Recall that

$$\frac{dy}{dx} \Big|_{(r, \theta)} = \frac{f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)}{f'(\theta) \cos(\theta) - f(\theta) \sin(\theta)}$$

$$r = f(\theta) = 2 + 2 \cos(\theta),$$

when $r = f(\theta)$.

$$f'(\theta) = -2 \sin(\theta)$$

$$\textcircled{1} \quad \theta = \frac{\pi}{6},$$

$$\frac{dy}{dx} \Big|_{\theta=\frac{\pi}{6}} = \frac{-2\left(\frac{1}{2}\right)^2 + \left(2 + 2\left(\frac{\sqrt{3}}{2}\right)\right) \cdot \frac{\sqrt{3}}{2}}{-2\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} - \left(2 + 2\left(\frac{\sqrt{3}}{2}\right)\right) \cdot \frac{1}{2}}$$

$$= -1 \quad (\text{I think})$$

$$x = r \cos(\theta) = \left(2 + \sqrt{3}\right) \cdot \frac{\sqrt{3}}{2} = \sqrt{3} + \frac{3}{2}$$

$$y = r \sin(\theta) = \left(2 + \sqrt{3}\right) \cdot \frac{1}{2} = 1 + \frac{\sqrt{3}}{2}$$

$$y - \left(1 + \frac{\sqrt{3}}{2}\right) = -1 \left(x - \left(\sqrt{3} + \frac{3}{2}\right)\right)$$

$$f'(\theta) = 2 \sin(\theta),$$

$$f'(\frac{\pi}{6}) = 2 \cdot \frac{1}{2} = 1$$

$$f(\frac{\pi}{6}) = 2 - 2\left(\frac{\sqrt{3}}{2}\right) = 2 - \sqrt{3}$$

$$\frac{dy}{dx} \Big|_{\theta=\frac{\pi}{6}} = \frac{1 \cdot \frac{1}{2} + \left(2 - \sqrt{3}\right) \cdot \frac{\sqrt{3}}{2}}{1 \cdot \frac{\sqrt{3}}{2} - \left(2 - \sqrt{3}\right) \cdot \frac{1}{2}}$$

$$= 1 \quad (\text{I think})$$

$$x = r \cos(\theta) = \left(2 - \sqrt{3}\right) \cdot \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{3}{2}$$

$$y = r \sin(\theta) = \left(2 - \sqrt{3}\right) \cdot \frac{1}{2} = 1 - \frac{\sqrt{3}}{2}$$

$$y - \left(1 - \frac{\sqrt{3}}{2}\right) = 1 \left(x - \left(\sqrt{3} - \frac{3}{2}\right)\right)$$

b) Find the length of each curve.

By symmetry, and some algebra

$$\begin{aligned} \int_0^{2\pi} \sqrt{(2+2\cos(\theta))^2 + (-2\sin(\theta))^2} d\theta &= 2 \int_0^{\pi} \sqrt{8+8\cos(\theta)} d\theta \\ &= 2 \int_0^{\pi} \sqrt{16\cos^2(\frac{\theta}{2})} d\theta \\ &= 8 \int_0^{\pi} \cos(\frac{\theta}{2}) d\theta \\ &= 16 \sin(\frac{\theta}{2}) \Big|_0^{\pi} \\ &= 16 \end{aligned}$$

(Again, we changed from $\int_0^{2\pi} \rightarrow 2 \int_0^{\pi}$

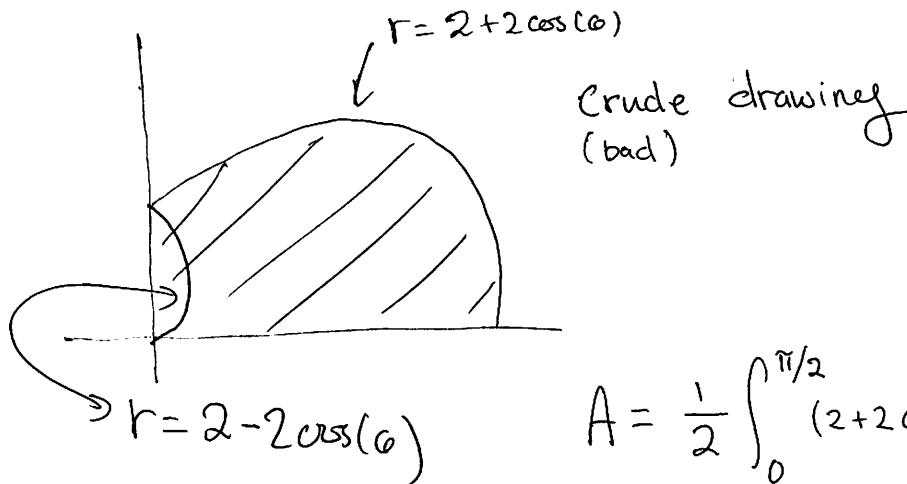
because $\sqrt{16\cos^2(\frac{\theta}{2})} = 4 |\cos(\frac{\theta}{2})|$ and integrating absolute value signs is a pain.)

By "visual inspection," these curves should have the same length, but let's check:

$$\begin{aligned} \int_0^{2\pi} \sqrt{(2-2\cos(\theta))^2 + (2\sin(\theta))^2} d\theta &= 2 \int_0^{\pi} \sqrt{8-8\cos(\theta)} d\theta \\ &= 2 \int_0^{\pi} \sqrt{16\sin^2(\frac{\theta}{2})} d\theta \\ &= 2 \int_0^{\pi} 4 \sin(\frac{\theta}{2}) d\theta \\ &= 8 (-2\cos(\frac{\theta}{2})) \Big|_0^{\pi} \\ &= 16 \end{aligned}$$

(now that I think about it, we don't need to do the $\int_0^{2\pi} \rightarrow 2 \int_0^{\pi}$ on this one...)

c) Find the area between the curves in the first quadrant.



$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/2} (2+2\cos(\theta))^2 - (2-2\cos(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} 4 + 8\cos(\theta) + 4\cos^2(\theta) - (4 - 8\cos(\theta) + 4\cos^2(\theta)) d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} 16 \cos(\theta) d\theta \\ &= 8 \sin(\theta) \Big|_0^{\pi/2} \\ &= 8 \end{aligned}$$