Math 243
Spring 2019
Practice Exam 2
Doomsday
Time Limit: Probably Not Enough

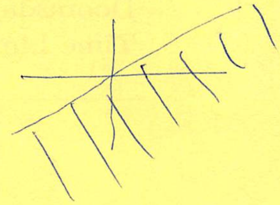
Name (Print): Solutions

Problem	Points	Score
1	0	
2	15	
3	20	
4	20	
5	20	
6	15	
7	15	
8	20	
9	20	
10	20	
Total:	165	

1. For the following functions, give the domain and range. Determine if the domain is open or closed (or neither), and determine if the domain is bounded or unbounded.

   a) $f(x, y) = \sqrt{x - y}$

   The domain is all points (x,y) where $x \geq y$:

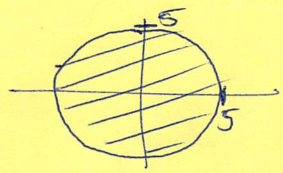
   it's unbounded and closed.

   The range is $[0, \infty)$.

   b) $g(x, y) = x^2 + y^2 - 3$, The Domain is $\mathbb{R}^2$ and is both open and closed, and unbounded.

   The Range is $[-3, \infty)$.

   c) $h(x, y) = \sqrt{25 - x^2 - y^2}$ The domain is $\{(x,y) \mid x^2+y^2 \leq 25\}$

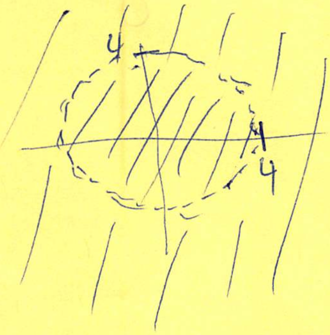
   The range is $[0, 5]$.

2. (15 points) Let $f(x,y) = \frac{1}{16-x^2-y^2}$

a) Find the domain and range of $f(x,y)$. Note: The range is tricky.

The domain is all (x,y) where $x^2+y^2 \neq 16$,

aka



. The range is $(-\infty, 0) \cup [\frac{1}{16}, \infty)$

b) Is the domain open/closed or neither? What is the boundary of the domain? Is the domain bounded or unbounded?

The domain is open. The boundary of the domain is $x^2+y^2=16$ and the domain is unbounded.

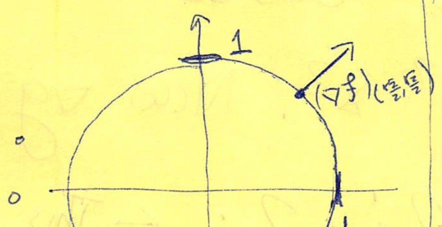
c) Graph the level curves $f(x,y) = \frac{1}{15}$ and $f(x,y) = 1$. Include the vector $\nabla f|_{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$ on the appropriate level curve.

$f(x,y) = \frac{1}{15}$

$\frac{1}{16-x^2-y^2} = \frac{1}{15}$

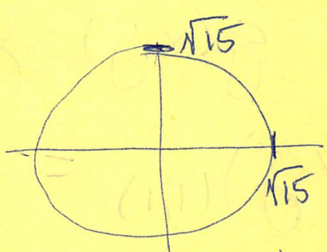
$15 = 16-x^2-y^2$

$x^2+y^2 = 1$



$f(x,y) = 1 \Rightarrow 1 = \frac{1}{16-x^2-y^2}$

$15 = x^2+y^2$



(not to scale)

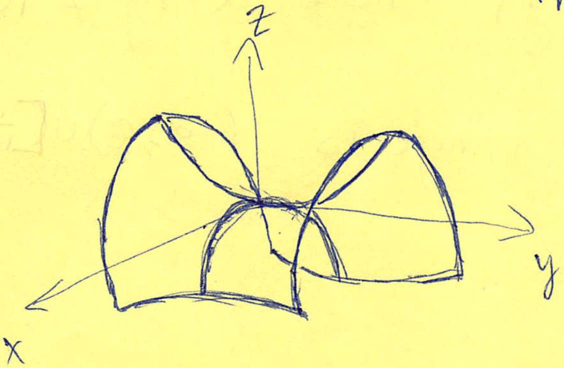
$\nabla f = \frac{2x}{(16-x^2-y^2)^2} i + \frac{2y}{(16-x^2-y^2)^2} j$, $(\nabla f)_{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$

$= \frac{\sqrt{2}}{15^2} i + \frac{\sqrt{2}}{15^2} j$

3. Consider the function $f(x, y, z) = y^2 - x^2 - z$.

(a) (10 points) Graph the level surface $f(x, y, z) = 0$.

This is $z = y^2 - x^2$ . . . .



(b) (10 points) Suppose we stood on this surface at the point $(1, 1, 0)$. The trail we are on goes directly up the surface in the "steepest" direction. What is the direction and how steep is the trail?

∇f usually points in the direction of the greatest rate of change, ~~So~~ ~~By~~ but f is a function of three variables . . . . so

set $g(x, y) = y^2 - x^2$. Now $\nabla g = -2x\,i + 2y\,j$

and $(\nabla g)_{(1,1)} = -2i + 2j$. ← This is the direction of the trail

$$\left(D_{\nabla g}\, g\right)_{(1,1)} = \left(\nabla g \cdot \frac{\nabla g}{|\nabla g|}\right)_{(1,1)} = |(\nabla g)_{(1,1)}| = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$= 2\sqrt{2}$ ← This is how steep the trail is

4. (a) (10 points) Find $\lim_{(x,y)\to(2,2)} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$ if it exists.

$$= \lim_{(x,y)\to(2,2)} \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) + 2(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(2,2)} = \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}+2)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(2,2)} \sqrt{x}+\sqrt{y}+2$$

$$= 2\sqrt{2}+2$$

(b) (10 points) Find $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$ if it exists.

Consider the path $y = mx^2$. Along this path, $\lim_{(x,y)\to(0,0)} = \lim_{x\to 0} \frac{x^4 - (mx^2)^2}{x^4 + (mx^2)^2}$

$$= \lim_{x\to 0} \frac{x^4(1-m^2)}{x^4(1+m^2)}$$

$$= \frac{1-m^2}{1+m^2}.$$

This depends on m, so, there are different paths that yeild different limits... meaning the limit does not exist.

5. (a) (10 points) Find $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{y^2+\sin(y)}{y^4}+x\right)\right)$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{y^2+\sin(y)}{y^4}+x\right)\right)$$

$$= \frac{\partial}{\partial y}(1)$$

$$= 0$$

(b) (10 points) Let $f(x,y,z) = \frac{ye^{xyz}}{x}$. Find f_x, f_y and f_z. Then, find f_{xyz}.

$$f_x = \frac{zy^2e^{xyz}\cdot x - ye^{xyz}}{x^2} = \frac{zy^2}{x}e^{xyz} - \frac{y}{x^2}e^{xyz}$$

$$f_y = \frac{e^{xyz} + yxze^{xyz}}{x} = \frac{e^{xyz}}{x} + yze^{xyz}$$

$$f_z = \frac{xy^2e^{xyz}}{x} = y^2e^{xyz}$$

$$f_{xz} = \frac{\partial}{\partial x}(f_z) = y^2e^{xyz}\cdot(yz) = y^3ze^{xyz}$$

$$f_{xyz} = \frac{\partial}{\partial y}(f_{xz}) = 3y^2ze^{xyz} + y^3ze^{xyz}\cdot xz$$
$$= 3y^2ze^{xyz} + xy^3z^2e^{xyz}$$

6. (15 points) a) Let $f(x,y)$ be a function with continuous partial derivatives. Suppose that $r(t) = g(t)\ i + h(t)\ j$ is a (differentiable) vector valued function and $f(g(t), h(t)) = c$ for some constant c. Show that ∇f and $\frac{dr}{dt}$ are orthogonal along this level curve.

Along this curve we have

$$f(g(t), h(t)) = C,$$

taking $\frac{d}{dt}$ of both sides we obtain

$$\underbrace{\frac{\partial f}{\partial x}\cdot\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt}}_{\nabla f \cdot \frac{dr}{dt}} = 0,$$

$$\nabla f \cdot \frac{dr}{dt} = 0.$$

Whence, ∇f and $\frac{dr}{dt}$ are orthogonal along this curve.

b) Find the the derivative of $f(x,y) = \ln(x^2+y^2)$ in the direction of $v = i + j$ at the point $(1,1)$.

$$\nabla f = \frac{1}{x^2+y^2}\cdot 2x\ i + \frac{1}{x^2+y^2}\cdot 2y\ j$$

$$(\nabla f)_{(1,1)} = i + j. \quad \frac{v}{|v|} = \frac{1}{\sqrt{2}}(i+j)$$

$$(D_v f)_{(1,1)} = (\nabla f)_{(1,1)} \cdot \frac{v}{|v|}$$

$$= \frac{1}{\sqrt{2}}\left((i+j)\cdot(i+j)\right)$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

7. (15 points) a) Let $z = x^2 - y^2 + 3$. Find the equation of the tangent plane at the point $(1, 1, 3)$.

Set $g(x,y,z) = x^2 - y^2 + 3 - z$, now $g = 0$ is the surface $z = x^2 - y^2 + 3$. $\nabla g = 2x\,i - 2y\,j - k$.

$\therefore$ The tangent plane has equation

$$2(x-1) - 2(y-1) - (z-3) = 0.$$

b) Let $f(x, y, z) = x^3 + y^2 + 3z + 4$. On the level surface $f(x, y, z) = 0$, give the equation of the tangent plane at the point $(1, 2, -3)$.

$\nabla f = 3x^2\,i + 2y\,j + 3k$,

$\therefore$ the tangent plane has equation

$$3(x-1) + 4(y-2) + 3(z+3) = 0.$$

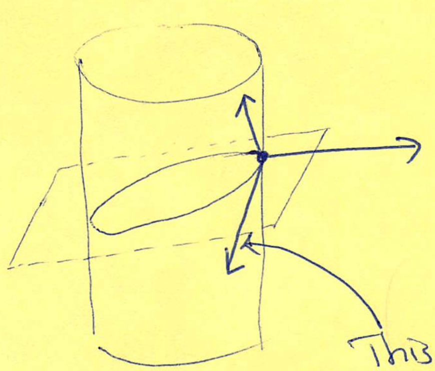
c) The surface $x^2 + y^2 = 4$ is "sliced" by the plane $x + y + z + 1 = 0$ and forms an ellipse. Find the parametric equations for the tangent line to this ellipse at the point $(2, 2, -5)$.

Let $g(x,y,z) = x^2 + y^2 - 4$, then

$\nabla g = 2x\,i + 2y\,j$, and $(\nabla g)_{(2,2,-5)} = 4i + 4j$

This is the vector, normal to the surface $g = 0$, at the point $(2,2,-5)$.

The vector $i + j + k$ is normal to the plane.



This vector is $(4i + 4j) \times (i + j + k)$

$$= \begin{vmatrix} i & j & k \\ 4 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \underline{4i - 4j}$$

So, parametric equations for the tangent line are

$x = 2 + 4t$, $y = 2 - 4t$, $z = -5$

8. (a) (20 points) Let $f(x,y) = 9x^3 + y^3/3 - 4xy$. Use the second derivative test to find any local min/max or saddle points.

$f_x = 27x^2 - 4y$ and $f_y = y^2 - 4x$. If $f_x = 0 = f_y$ we have critical points $(0,0)$ and $(\frac{4}{9}, \frac{4}{3})$:

$f_y = 0 \Rightarrow x = \frac{y^2}{4}$, and this combined w/

$f_x = 0$ gives $0 = 27x^2 - 4y = 27\left(\frac{y^2}{4}\right)^2 - 4y = \left(\frac{27}{64}y^3 - 1\right)4y$

so, $y = 0$ or $y = \frac{4}{3}$. Since $x = y^2/4$, we obtain said critical points.

Computing $f_{xx} = 54x$ and $f_{yy} = 2y$ we see that $D = f_{xx}f_{yy} - (f_{xy})^2 = 108xy - 16$, and

$D(0,0) < 0$ so $(0,0)$ is a saddle point

and $D\left(\frac{4}{9}, \frac{4}{3}\right) = 108 \cdot \left(\frac{4}{9}\right)\left(\frac{4}{3}\right) - 16$

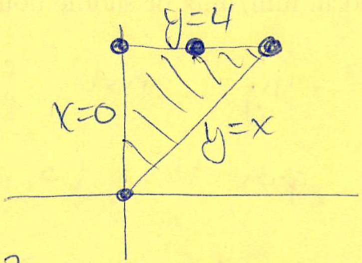
$= 9 \cdot 12 \left(\frac{4}{9}\right)\left(\frac{4}{3}\right) - 16$

$= 4^3 - 16 > 0$

along w/ $f_{xx}\left(\frac{4}{9}, \frac{4}{3}\right) = 54\left(\frac{4}{9}\right) = 24 > 0$ show that $\left(\frac{4}{9}, \frac{4}{3}\right)$ is a local minimum.

9. (20 points) Find the absolute maxima and minima of $f(x,y) = x^2 - xy + y^2 + 1$ on the closed region bounded by $x = 0$, $y = 4$ and $y = x$.

First let's draw the region:



For critical points we have

$f_x = 2x - y$ and $f_y = -x + 2y$.

If $f_x = 0$, then $y = 2x$. If $f_y = 0$, then $0 = -x + 4x$ and so $x = 0$ and further that $y = 0$. So, $(0,0)$ is our only C.P.

Boundary: for $x=0$: $f(0,y) = y^2 + 1$, $f'(0,y) = 2y$

So we get the point $(0,0)$ (again).

We also pick-up $(0,4)$.

for $y = 4$, $f(x,4) = x^2 - 4x + 17$, $f'(x,4) = 2x - 4 \Rightarrow (2,4)$.

We also get the point $(4,4)$.

for $y = x$, $f(x,x) = x^2 - x^2 + x^2 + 1 = x^2 + 1$,

$f'(x,x) = 2x \Rightarrow (0,0)$ (again).

$f(0,0) = 1$, $f(0,4) = 17$, $f(4,4) = 17$, $f(2,4) = 13$.

~~Thus, (0,0)~~

Thus, at $(0,0)$, f has an abs. min and at $(0,4)$ and $(4,4)$ f has an abs max.

10. (20 points) Find the point on the plane

$$2x + 2y + 2z = 2$$

which is closest to the origin.

We aim to minimize the function

$$f(x,y,z) = x^2+y^2+z^2$$

subject to the constraint, $x+y+z=1$.

Now, $f(x,y) = x^2+y^2+(1-x-y)^2$.

$f_x = 2x + 2(1-x-y)(-1) = 4x+2y-2$

and $f_y = 2y + 2(1-x-y)(-1) = 4y+2x-2$

If $f_x = 0$ then $y = 1-2x$, and this w/ $f_y = 0$ gives

$$0 = 4(1-2x)+2x-2$$
$$= 2-6x$$

and thus $x = 1/3$. Using $y = 1-2x$, this gives $y = 1/3$. So, the only c.p. of our function is $(\frac{1}{3}, \frac{1}{3})$.

$f_{xx} = 4$ and $f_{yy} = 4$, $f_{xy} = 2$. This information tells us that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the closest point on the plane to the origin.