Math 243
Spring 2019
Practice Exam 1
Doomsday
Time Limit: Probably Not Enough

Name (Print): Solutions

Problem	Points	Score
1	15	
2	30	
3	40	
4	15	
5	20	
6	20	
7	20	
8	10	
9	10	
10	25	
11	40	
Total:	245	

1. (15 points) Let $P = (1, 2, 3)$ and $Q = (0, -1, 2)$.

a) Find the distance between P and Q.

$|\vec{PQ}| = |\vec{Q} - \vec{P}| = \sqrt{(-1)^2 + (-3)^2 + (-1)^2} = \sqrt{11}$

b) Give the equation of a sphere, centered at P, that has the point Q on its surface.

$(x - 1)^2 + (y-2)^2 + (z-3)^2 = 11$

c) Find the vector $\overrightarrow{PQ}$.

$\vec{PQ} = \vec{Q} - \vec{P} = \langle -1, -3, -1 \rangle$

d) Find the angle between $\vec{P}$ and $\vec{Q}$.

Recall that $\theta = \cos^{-1}\left(\frac{\vec{P} \cdot \vec{Q}}{\|P\| \|Q\|}\right)$.

$|P| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$|Q| = \sqrt{0^2 + (-1)^2 + (2)^2} = \sqrt{5}$

$P \cdot Q = 1 \cdot 0 + 2(-1) + 3 \cdot 2 = 4$

$\theta = \cos^{-1}\left(\frac{4}{\sqrt{14}\sqrt{5}}\right)$

e) Parametrize (with parametric equations) the line **segment** which starts at P and ends at Q.

$0 \leq t \leq 1$

$\vec{P}(1-t) + \vec{Q}t = \langle (1-t) + 0t, 2(1-t) - t, 3(1-t) + 2t \rangle$

$= \langle 1 - t, 2 - 3t, 3 - t \rangle$

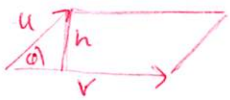
Parametric equations: $x = 1 - t$, $y = 2 - 3t$, $z = 3 - t$ for $0 \leq t \leq 1$

2. The following points define the vertices of a triangle:

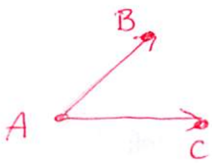
$$A = (0,0,1) \quad B = (1,3,-1) \quad C = (2,2,2)$$

(a) (10 points) Find the area of the triangle.

Recall:



Area of parallelogram is $|v||u||\sin(\theta)|$, and $|u \times v| = |u||v||\sin(\theta)|$. The area of the triangle is $\frac{1}{2}$ the parallelogram.



$\vec{AB} = \langle 1,3,-2\rangle$, $\vec{AC} = \langle 2,2,1\rangle$, $\vec{AB} \times \vec{AC} = 7i - 5j - 4k$

So, the area of the triangle is $\frac{3\sqrt{10}}{2}$.

(b) (10 points) Find an equation of the plane which contains the triangle.

Using $\vec{AB} \times \vec{AC}$ as our normal vector:

$$7(x-0) - 5(y-0) - 4(z-1) = 0.$$

Note: You can also use the vector $\vec{AC} \times \vec{AB}$ as a normal vector, also, any of the points A, B, or C will work.

(c) (10 points) Find the distance between the point $\underbrace{(3,2,1)}_{P}$ and the plane from part b).

$\vec{AP} = \langle 3,2,0\rangle$, $d = |\mathrm{Proj}_{\hat{n}} \vec{AP}| = \dfrac{|\vec{n} \cdot \vec{AP}|}{|\vec{n}|}$

$$= \frac{11}{3\sqrt{10}}$$

3. (a) (10 points) Let $u = i + j$, $v = i + j + k$. Find the projection of u onto $u + v$.

$$\text{Proj}_{u+v} u = \frac{4}{9}(2i + 2j + k)$$

(b) (10 points) The vectors $u = i + 2j$ and $v = j + 3k$ lie in a plane that goes through the point $P = (1, 0, 1)$. Give the equation of this plane.

$u \times v = 6i - 3j + k$,

Plane: $6(x-1) - 3(y-0) + 1(z-1) = 0$

(c) (10 points) Give the equation of a line, perpendicular to the plane $2x + 3y + z = 6$, that goes through the point $(1, 0, 1)$.

A normal vector for this plane is $\langle 2,3,1 \rangle$ and we can use this as the "slope" of our line:

$x = 1 + 2t$, $y = 0 + 3t$, $z = 1 + t$

(d) (10 points) Find the point in space the line from part c) intersects the plane from part b).

I hope you can figure this one out.....

4. (15 points) Match the surface with its equation

$$1 = x^2 + y^2 - z^2$$ f)

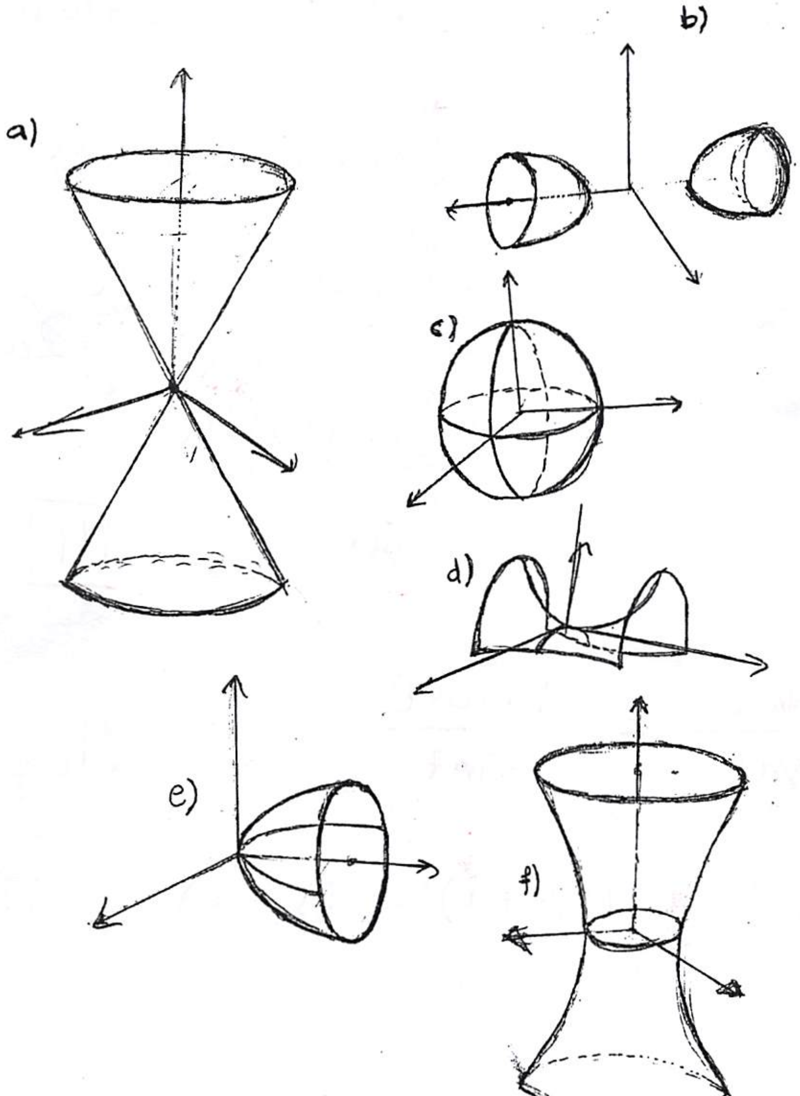
$$y = x^2 + z^2$$ e)

$$1 = x^2 + y^2 + z^2$$ c)

$$z = y^2 - x^2$$ d)

$$0 = x^2 + y^2 - z^2$$ a)

$$x^2 = z^2 + y^2 + 1$$ b)



5. (20 points) Find the length of the curve given by the parametric equations

$$x = \cos(t) \quad \text{and} \quad y = t + \sin(t) \quad \text{for } 0 \le t \le \pi.$$

Also, find the equation of the tangent line when $t = \frac{\pi}{2}$.

The length of our curve is

$$\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi} \sqrt{(-\sin(t))^2 + (1+\cos(t))^2}\, dt$$

$$= \int_0^{\pi} \sqrt{2 + 2\cos t}\, dt$$

$$= 2\int_0^{\pi} |\cos(t/2)|\, dt$$

$\cos(t/2) \geq 0$ on $[0, \pi]$.

$$= 2\int_0^{\pi} \cos(t/2)\, dt$$

$$= 4\sin(t/2)\Big|_{t=0}^{t=\pi} = \boxed{4}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+\cos t}{-\sin t}, \quad \text{so } \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$$

E.O.T.L.: $y - (\pi/2 + 1) = -(x - 0)$

6. (20 points) Find the length of the curve

$$x = \frac{y^3}{6} + \frac{1}{2y}$$

from $y = 2$ to $y = 3$.

$$x = \frac{t^3}{6} + \frac{1}{2t}, \quad y = t \quad \text{for} \quad 2 \le t \le 3$$

The length of the curve is

$$\int_2^3 \sqrt{\left(\frac{t^2}{2} - \frac{1}{2t^2}\right)^2 + 1^2}\, dt = \int_2^3 \sqrt{\frac{t^4}{4} - \frac{1}{2} + \frac{1}{4t^4} + 1}\, dt$$

$$= \int_2^3 \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2}\, dt$$

$$= \int_2^3 \frac{t^2}{2} + \frac{1}{2t^2}\, dt$$

Say a prayer for the abs. value.

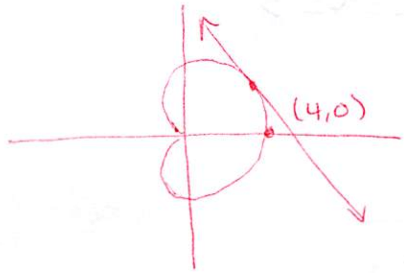
$$= \frac{t^3}{6} - \frac{1}{2t}\Bigg|_{t=2}^{t=3}$$

$$= 13/4$$

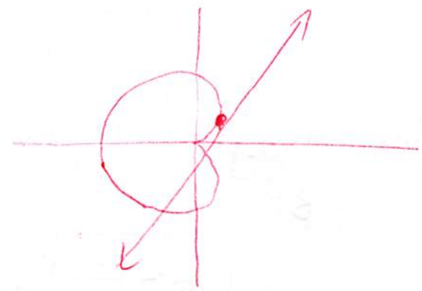
7. (20 points) Consider the polar coordinate equations $r = 2(1+\cos(\theta))$ and $r = 2(1-\cos(\theta))$.

a) Graph both of these curves. For both graphs, find and plot the equation of the tangent line when $\theta = \dfrac{\pi}{6}$.

$r = 2(1+\cos\theta)$



$r = 2(1-\cos\theta)$



Recall that $\left.\dfrac{dy}{dx}\right|_{(r,\theta)} = \dfrac{f'(\theta)\sin(\theta)+f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta)-f(\theta)\sin(\theta)}$

when $r = f(\theta)$.

$f'(\theta) = -2\sin\theta$

$\left.\dfrac{dy}{dx}\right|_{\theta=\pi/6} = -1$ (after some simp)

$x = (2+\sqrt{3})\cdot\dfrac{\sqrt{3}}{2} = \sqrt{3}+3/2$

$y = (2+\sqrt{3})\cdot\dfrac{1}{2} = 1+\dfrac{\sqrt{3}}{2}$

Equation of tangent line:

$y-\left(1+\dfrac{\sqrt{3}}{2}\right) = -1\left(x-\left(\sqrt{3}+\dfrac{3}{2}\right)\right)$

$f'(\theta) = 2\sin(\theta)$

$\left.\dfrac{dy}{dx}\right|_{\theta=\pi/6} = 1$ (after some simp)

$x = (2-\sqrt{3})\dfrac{\sqrt{3}}{2} = \sqrt{3}-\dfrac{3}{2}$

$y = (2-\sqrt{3})\cdot\dfrac{1}{2} = 1-\dfrac{\sqrt{3}}{2}$

$y-\left(1-\dfrac{\sqrt{3}}{2}\right) = 1\left(x-\left(\sqrt{3}-\dfrac{3}{2}\right)\right)$

Recall that

$x = r\cos(\theta) = f(\theta)\cos(\theta)$

$y = r\sin(\theta) = f(\theta)\sin(\theta)$

b) Find the length of each curve.

To get around absolute value, we can use symmetry

$$\int_0^{2\pi} \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{8+8\cos\theta}\, d\theta$$

$$= 8\int_0^{\pi} \cos(\theta/2)\, d\theta$$

$$= 16 \sin(\theta/2)\Big|_0^{\pi}$$

$$= 16$$

By looking both curves, I'm expecting them to have the same length.

$$\int_0^{2\pi} \sqrt{(2-2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{8-8\cos\theta}\, d\theta$$

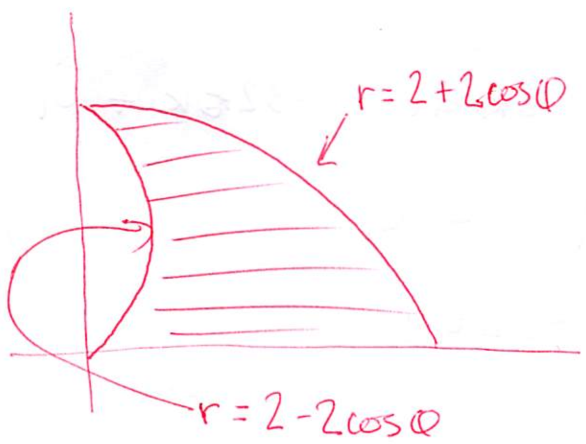
$$= 2\int_0^{\pi} \sqrt{16\sin^2(\tfrac{\theta}{2})}\, d\theta$$

$$= 2\int_0^{\pi} 4\sin(\tfrac{\theta}{2})\, d\theta$$

$$= 8\left(-2\cos(\theta/2)\Big|_0^{\pi}\right)$$

$$= 16$$

c) Find the area between the curves in the first quadrant.



$$A = \frac{1}{2}\int_0^{\pi/2} (2+2\cos\theta)^2 - (2-2\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 16\cos(\theta)\, d\theta = 8\sin\theta\Big|_0^{\pi/2} = 8$$

8. (10 points) Show that if $r(t)$ is a differentiable vector valued function and $|r(t)| = C$ for a constant C, then $r(t)$ and $\frac{dr}{dt}$ are orthogonal.

Recall that $v \cdot v = |v|^2$. We have $c^2 = |r(t)|^2 = r(t) \cdot r(t)$.

Now,

$$0 = \frac{d}{dt}(c^2) = \frac{d}{dt}(r(t) \circ r(t))$$
$$= \frac{dr}{dt} \circ r(t) + r(t) \circ \frac{dr}{dt}$$
$$= 2\frac{dr}{dt} \circ r(t)$$

and so $0 = \frac{dr}{dt} \circ r(t)$, $\therefore$ $r(t)$ and $\frac{dr}{dt}$ are orthogonal.

9. (10 points) Find $r(t)$ if

$$\frac{d^2r}{dt^2} = -32k, \quad r(0) = 100k, \quad \left.\frac{dr}{dt}\right|_{t=0} = 8\,i + 8\,j$$

$$\frac{dr}{dt} = \int \frac{d^2r}{dt^2}\,dt = \int -32k\,dt = -32\cancel{t}k + \vec{C}_1$$

$$8i + 8j = \left.\frac{dr}{dt}\right|_{t=0} = \vec{C}_1$$

$$\therefore \frac{dr}{dt} = 8i + 8j - 32tk$$

$$r(t) = \int \frac{dr}{dt}\,dt = \int 8i + 8j \cancel{+} -32tk\,dt$$
$$= 8ti + 8tj - 16t^2k + \vec{C}_2$$

$$100k = r(0) = \vec{C}_2,$$

$$\therefore r(t) = 8ti + 8tj + (100 - 16t^2)k$$

10. Let $r(t) = t\cos(t)\ i + t\sin(t)\ j + \frac{2\sqrt{2}}{3}t^{3/2}\ k$.

(a) (10 points) Find the parametric equations of the tangent line to the curve when $t = \frac{\pi}{3}$.

$$r'(t) = (\cos t - t\sin t)\,i + (\sin t + t\cos t)\,j + \sqrt{2}\sqrt{t}\,k$$

$$r'(\pi/3) = \left(\frac{1}{2} - \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2}\right)i + \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\cdot\frac{1}{2}\right)j + \sqrt{2}\cdot\sqrt{\pi/3}\ k$$

$$r(\pi/3) = \frac{\pi}{3}\cdot\frac{1}{2}\,i + \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2}\,j + \frac{2\sqrt{2}}{3}\left(\frac{\pi}{3}\right)^{3/2}k$$

$$\therefore\quad x = \frac{\pi}{6} + \left(\frac{3-\sqrt{3}\pi}{6}\right)t,\quad y = \frac{\sqrt{3}\pi}{6} + \left(\frac{3\sqrt{3}+\pi}{6}\right)t,\quad z = \frac{2\sqrt{2}}{3}(\pi/3)^{3/2} + \sqrt{\frac{2\pi}{3}}\,t$$

(b) (15 points) Find the length of the curve from $t = 0$ to $t = \pi$.

Length of the curve is

$$\int_0^\pi \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (\sqrt{2}\sqrt{t})^2}\,dt$$

$$= \int_0^\pi \sqrt{t^2 + 2t + 1}\,dt$$

$$= \int_0^\pi \sqrt{(t+1)^2}\,dt$$

$$= \int_0^\pi t + 1\,dt \qquad (t+1 \geq 0 \text{ on } [0,\pi])$$

$$= \frac{t^2}{2} + t\Big|_0^\pi$$

$$= \frac{\pi^2}{2} + \pi$$

11. (40 points) For numbers $a, b \geq 0$, let

$$r(t) = a\cos(t)\ i + a\sin(t)\ j + bt\ k.$$

Find the unit tangent vector, T, the principle unit normal vector, N, the curvature κ, the unit binormal B, and the torsion τ of this curve. Give the equation of the osculating plane at $t = 0$.

$V(t) = r'(t) = -a\sin t\ i + a\cos t\ j + b\ k$, $|V| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$

$$T = \frac{V}{|V|} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin t\ i + a\cos t\ j + b\ k\right)$$

$$\frac{dT}{dt} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\cos t\ i - a\sin t\ j\right), \quad \left|\frac{dT}{dt}\right| = \sqrt{\frac{a^2}{a^2+b^2}\cos^2 t + \frac{a^2}{a^2+b^2}\sin^2 t}$$

$$= \frac{a}{\sqrt{a^2+b^2}}$$

$$N = \frac{dT/dt}{|dT/dt|} = \frac{\sqrt{a^2+b^2}}{a}\left(\frac{1}{\sqrt{a^2+b^2}}\left(-a\cos t\ i - a\sin t\ j\right)\right) = -\cos t\ i - \sin t\ j$$

$$\kappa = \frac{1}{|V|}\left|dT/dt\right| = \frac{1}{\sqrt{a^2+b^2}} \cdot \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ \frac{-a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{b\sin t}{\sqrt{a^2+b^2}}\ i - \frac{b\cos t}{\sqrt{a^2+b^2}}\ j + \frac{a}{\sqrt{a^2+b^2}}\ k$$

$$\frac{dB}{dt} = \frac{b\cos t}{\sqrt{a^2+b^2}}\ i + \frac{b\sin t}{\sqrt{a^2+b^2}}\ j, \quad \tau = \frac{-1}{|V|}\left(\frac{dB}{dt} \cdot N\right) = \frac{-1}{\sqrt{a^2+b^2}}\left(\frac{-b\cos^2 t}{\sqrt{a^2+b^2}} - \frac{b\sin^2 t}{\sqrt{a^2+b^2}}\right)$$

$$= \frac{b}{(\sqrt{a^2+b^2})^2} = \frac{b}{a^2+b^2}$$

$r(0) = a i$ and $B(0) = \frac{1}{\sqrt{a^2+b^2}}(-bj + ak)$, so the osculating plane has equation

$$-b(y-0) + a(z-0) = 0$$