

Math 243

Spring 2019

Final

Please Study

Time Limit: 120 minutes

No Notes

No Calculators

No Funny Business

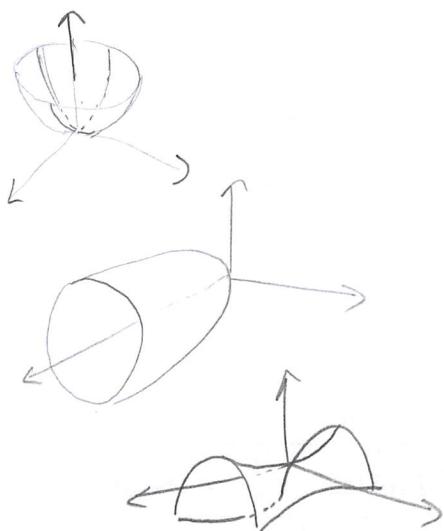
Name (Print):

Solutions

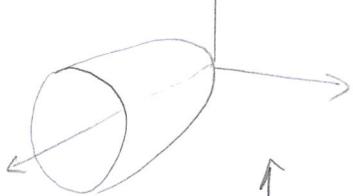
Problem	Points	Score
1	20	
2	25	
3	15	
4	10	
5	10	
6	10	
7	20	
8	40	
9	20	
10	20	
11	10	
12	20	
13	20	
14	15	
15	15	
Total:	270	

1. (a) (20 points) Sketch a graph of the following:

$$z = x^2 + y^2$$



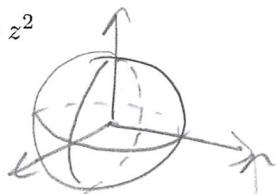
$$x = y^2 + z^2$$



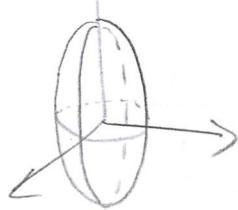
$$z = x^2 - y^2$$



$$9 = x^2 + y^2 + z^2$$



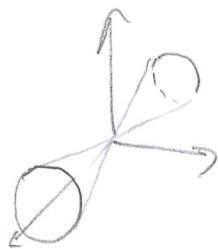
$$1 = \frac{z^2}{9} + x^2 + y^2$$



$$z^2 = x^2 + y^2$$



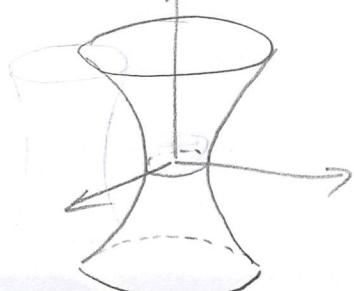
$$x^2 = z^2 + y^2$$



$$z^2 = x^2 + y^2 + 1$$



$$z^2 = x^2 + y^2 - 1$$



2. Let $P = (1, 2, 0)$, $Q = (1, 1, 3)$ and $R = (0, 0, 1)$.

- (a) (5 points) Find the angle between \vec{P} and \vec{Q} .

$$\theta = \cos^{-1} \left(\frac{\vec{P} \cdot \vec{Q}}{|\vec{P}| |\vec{Q}|} \right) = \cos^{-1} \left(\frac{3}{\sqrt{5} \sqrt{11}} \right)$$

- (b) (5 points) Parametrize the line segment from P to Q .

The vector \vec{PQ} is $\langle 0, -1, 3 \rangle$, so the segment is

$$x = 1, y = 2 - t, z = 3t \quad 0 \leq t \leq 1$$

- (c) (5 points) Give an equation of the plane containing P , Q and R .

$$\vec{PQ} \times \vec{PR} = \langle 0, -1, 3 \rangle \times \langle -1, -2, 1 \rangle = \begin{vmatrix} i & j & k \\ 0 & -1 & 3 \\ -1 & -2 & 1 \end{vmatrix} = 5i - (3)j - k$$

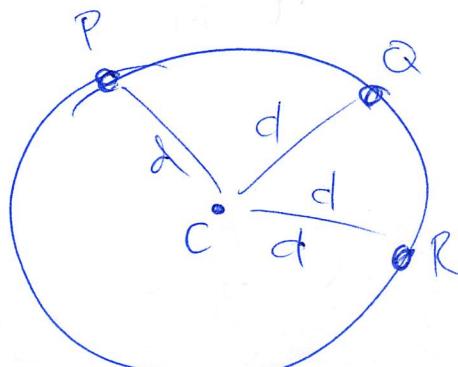
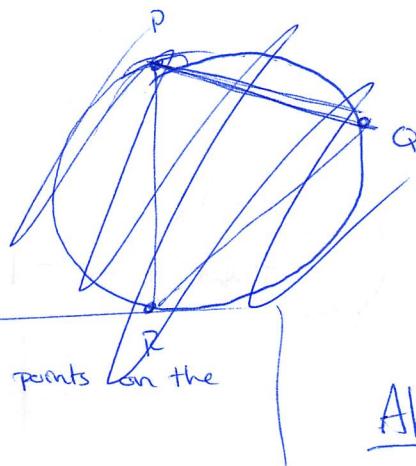
$$5(x-1) - 3(y-2) - z = 0$$

- (d) (5 points) Give the area of the triangle whose vertices are P , Q and R .

$$A_{\Delta} = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{\sqrt{5^2 + 9 + 1}}{2} = \frac{\sqrt{35}}{2}$$

- (e) (5 points) Give the equation of the sphere whose surface contains the points P , Q and R .

Hint:



Extra Credit:

Give all spheres w/ these points on the surface.

Also: P, Q, R and C are all in the same plane.

3. Consider the parametric equations

$$x = \cos(2t) \quad \text{and} \quad y = 2t + \sin(2t) \quad \text{for } 0 \leq t \leq \pi.$$

(a) (10 points) Find the length of this curve.

$$\begin{aligned} L &= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^\pi \sqrt{4\sin^2(2t) + (2 + 2\cos(2t))^2} dt \\ &= \int_0^\pi \sqrt{8 + 8\cos(2t)} dt \\ &= 4 \int_0^\pi |\cos(t)| dt \\ &= 4 \left[\int_0^{\pi/2} \cos(t) dt + \int_{\pi/2}^{\pi} -\cos(t) dt \right] \\ &= 8 \end{aligned}$$

(b) (5 points) Find the equation of the tangent line when $t = \frac{\pi}{4}$.

$$m = \frac{dy/dt}{dx/dt} = \frac{2 + 2\cos(2t)}{-2\sin(2t)}$$

$$m \Big|_{t=\pi/4} = \frac{2 + 2(0)}{-2(1)} = -1$$

Tangent Line $y - \left(\frac{\pi}{2} + 1\right) = -1(x - 0)$

4. (a) (5 points) Find $\lim_{(x,y) \rightarrow (4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$ if it exists.

$$\begin{aligned} &= \lim_{(x,y) \rightarrow (4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1} \cdot \left(\frac{\sqrt{x} + \sqrt{y+1}}{\sqrt{x} + \sqrt{y+1}} \right) \\ &= \lim_{(x,y) \rightarrow (4,3)} \frac{1}{\sqrt{x} + \sqrt{y+1}} = \frac{1}{4} \end{aligned}$$

(b) (5 points) Find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y}$ if it exists.

$$\text{let } y = mx^2, \text{ then } \frac{x^2}{x^2 + y} = \frac{x^2}{x^2(1+m)} = \frac{1}{1+m}$$

So, $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y} = \frac{1}{1+m}$ along this path. \therefore The limit DNE.

5. (10 points) For differentiable vector valued functions $u(t)$ and $v(t)$, prove that $\frac{d}{dt} \left(u(t) \cdot v(t) \right) = \frac{du}{dt} \cdot v(t) + u(t) \cdot \frac{dv}{dt}$

Hence we are to assume that

$$u(t) = f_1 i + g_1 j \quad \text{and} \quad v(t) = f_2 i + g_2 j.$$

$$u \cdot v = f_1 f_2 + g_1 g_2, \text{ so } \frac{d}{dt}(u \cdot v) = \underbrace{f'_1 f_2}_{\frac{du}{dt} \cdot v} + \underbrace{f_1 f'_2}_{v \cdot \frac{dv}{dt}} + \underbrace{g'_1 g_2}_{\frac{du}{dt} \cdot v} + \underbrace{g_1 g'_2}_{v \cdot \frac{dv}{dt}}$$

$$\text{So, } \frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}.$$

6. (10 points) Show that if $r(t)$ is a differentiable vector valued function and $|r(t)| = C$ for a constant C , then $r(t)$ and $\frac{dr}{dt}$ are orthogonal.

Notice that $r(t) \cdot r(t) = |r(t)|^2 = C^2$

By the above (#5)

$$0 = \frac{d}{dt}(C^2) = \frac{d}{dt}(r \cdot r) = \frac{dr}{dt} \cdot r + r \cdot \frac{dr}{dt} \\ = 2 \frac{dr}{dt} \cdot r(t)$$

So $\frac{dr}{dt} \cdot r(t) = 0 \therefore \text{they are orthogonal.}$

7. (20 points) Find $r(t)$ if

$$\frac{d^2r}{dt^2} = i + j - 32k, \quad \left. \frac{dr}{dt} \right|_{t=0} = 8i + 8j \quad \text{and} \quad r(0) = \cancel{2i + jk}$$

$$2i + j + k$$

$$\frac{dr}{dt} = t i + t j - 32 t k + \vec{C}_1$$

$$8i + 8j = r'(0) = \vec{C}_1 \quad \therefore \quad r'(t) = (t+8)i + (t+8)j - 32t k$$

$$r(t) = \int r'(t) dt = \left(\frac{t^2}{2} + 8t \right) i + \left(\frac{t^2}{2} + 8t \right) j - 16t^2 k + \vec{C}_2$$

$$2i + j + k = r(0) = \vec{C}_2, \quad \therefore$$

$$r(t) = \left(\frac{t^2}{2} + 8t + 2 \right) i + \left(\frac{t^2}{2} + 8t + 1 \right) j + (-16t^2 + 1) k$$

8. (40 points) Let

$$r(t) = \cos^3(t) \ i + \sin^3(t) \ j.$$

Find the unit tangent vector, T , the principle unit normal vector, N , the curvature, κ , the unit binormal, B , and the torsion, τ , of this curve.

Life is too short.
let's kill this problem on the
final. meaning: there won't be any
questions about it.

9. (20 points) Let $f(x, y, z) = \frac{z^2 e^{zxy}}{x}$. Find f_x, f_y, f_z and f_{xyz} .

$$f_x = \frac{yz^3 e^{zxy} x - z^2 e^{zxy}}{x^2}$$

$$f_y = \frac{z^2}{x} e^{zxy} \cdot xz = z^3 e^{zxy}$$

$$f_z = \frac{z^2 e^{zxy} \cdot xy + 2z e^{zxy}}{x}$$

$$f_{xy} = yz^4 e^{zxy}$$

$$f_{xyz} = 4z^3 y e^{zxy} + yz^4 e^{zxy} \cdot xy$$

10. (20 points) a) Suppose that $r(t) = g(t)i + h(t)j$ is a vector valued function such that $f(g(t), h(t)) = c$ for some constant c . Show that ∇f and $\frac{dr}{dt}$ are orthogonal along this level curve.

Since $f(g(t), h(t)) = c$, we

have $\frac{d}{dt}(f(g(t), h(t))) = \frac{d}{dt}(c)$

$$\frac{\partial f}{\partial x} \frac{dy}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j \right) \cdot \left(\frac{dy}{dt} i + \frac{dy}{dt} j \right) = 0$$

$$\nabla f \cdot \frac{dr}{dt} = 0$$

So, ∇f and $\frac{dr}{dt}$ are orthogonal.

11. (a) (5 points) Let $f(x, y) = x^2 - y^2 + 3$. Find an equation of the tangent plane at the point $(4, 4, 3)$.

~~Define $g(x, y, z) = x^2 - y^2 + 3 - z$,~~ Define $g(x, y, z) = x^2 - y^2 + 3 - z$,
Now, $g=0$ gives $f(x, y)$ as the graph.

$\nabla g = 2xi - 2yj - k$ so, the tangent
plane has equation

$$\boxed{8(x-4) - 8(y-4) - (z-3) = 0}$$

- (b) (5 points) Let $f(x, y, z) = 2x^3 + 4y^2 - z^2$. Verify that the point $(1, 1, 1)$ is on the level surface $f = 5$, then find an equation of the tangent plane at that point.

~~$$\nabla f = 6x^2i + 8yj - 2zk,$$~~

~~Equation of plane:~~

~~$$6(x-1) + 8(y-1) + 2(z-1) = 0$$~~

Note: $f(1, 1, 1) = 2 + 4 - 1 = 5$,

so, YES! this point is on the
level surface!

12. Let $f(x, y) = \frac{1}{\sqrt{1-x^2-y^2}}$.

- (a) (5 points) Find the domain and range of $f(x, y)$.

$D: \cancel{\text{Region}} \quad \underline{x^2+y^2 \geq 1}$

$R: [1, \infty)$

$D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 8\}$

- (b) (5 points) Is the domain open/closed or neither? What is the boundary of the domain?
Is the domain bounded or unbounded?

$x^2+y^2=1$

not in the domain.

- (c) (5 points) Graph the level curve $f(x, y) = 8$. Determine if $(\frac{3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}})$ is on this level curve.
If it is, plot ∇f on the level curve at this point.

$(\frac{\sqrt{63}}{16}, \frac{\sqrt{63}}{16})$ would be
a better point
to use.

lets just kill this problem.

- (d) (5 points) Find find a c such that the level curve $f(x, y) = c$ contains the point $(\frac{1}{2}, \frac{1}{\sqrt{2}})$.

$$f\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{1-\frac{1}{4}-\frac{1}{2}}} = \frac{1}{\sqrt{\frac{1}{4}}} = 2$$

13. (a) (20 points) Let $f(x, y) = 8x^3 + y^3 + 6xy$. Use the second derivative test to find any local min/max or saddle points. You do not need to evaluate $f(x, y)$ at these points.

$$f_x = 24x^2 + 6y \quad \text{and} \quad f_y = 3y^2 + 6x$$

If $f_x = 0$, then $y = -4x^2$. Now, if $f_y = 0$,

$$0 = 3(-4x^2)^2 + 6x$$

$$0 = 3 \cdot 2^4 x^4 + 6x$$

$$= 6x(2^3 x^3 + 1) \Rightarrow x = 0 \text{ (and so } (0, 0) \text{ is a c.p.)}$$

$$\text{or } x = \frac{-1}{\cancel{2}} \text{ so, } y = -4 \left(\frac{-1}{\cancel{2}}\right)^2$$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$= 48x \cdot 9y - 36$$

$$= \cancel{6} \cancel{6} - 1$$

$$= \cancel{6} \cancel{6} - 1$$

$$\text{so, c.p. is } \left(\frac{-1}{2}, -1\right), \left(\frac{-1}{2}, \frac{1}{2}\right).$$

~~Both R.P.'s~~
~~are saddle points~~

$$f_{xx}\left(\frac{-1}{2}\right) = -24, \text{ so } (0, 0) \text{ is a saddle}$$

and $(-\frac{1}{2}, -1)$ is a local max!

14. (15 points) Find the **cubic** (degree 3) approximation for the function

$$f(x, y) = e^{2x} \ln(1 + 3y)$$

centered at the origin.

$$f_x = 2e^{2x} \ln(1 + 3y), \quad f_{xx} = 4e^{2x} \ln(1 + 3y),$$

$$f_y = \frac{e^{2x}}{1 + 3y} \cdot 3, \quad f_{yy} = \frac{-9e^{2x}}{(1 + 3y)^2},$$

$$f_{xy} = \frac{2e^{2x}}{1 + 3y} \cdot 3, \quad f_{yyx} = \frac{-18e^{2x}}{(1 + 3y)^2},$$

$$f_{xxx} = 8e^{2x} \ln(1 + 3y)$$

$$f_{yyy} = \frac{54e^{2x}}{(1 + 3y)^3}$$

$$f_{xxy} = \frac{12e^{2x}}{1 + 3y},$$

$$f(x, y) \approx 3y + 6xy + -\frac{9}{2}y^2 - \frac{18}{2}y^2x + \frac{12}{2}x^2y + \frac{54}{3!}y^3$$

15. (15 points) There will be a "Lagrange" type question similar to ones found on the homework or worksheet. Here is one I took from the book:

Suppose that the temperature (in degrees Celsius) on the sphere $x^2 + y^2 + z^2 = 1$ is given by the function $T(x, y, z) = xyz^2$. Find the hottest and coldest points on the sphere. Would water freeze at any point on the sphere?

Let $g = x^2 + y^2 + z^2 - 1$, then $g = 0$ is our constraint! If $\nabla T = \lambda \nabla g$,

1) 2) 3)

then $y z^2 = \lambda 2x$, $x z^2 = \lambda 2y$ and $2xyz = \lambda 2z$. By inspection,

T is not largest or smallest if x, y or $z = 0$, so

let's assume they are not.

By 3) we have $\lambda = xy$. This w/ 2)

gives $xz^2 = 2xy^2$ and also w/ 3)

gives $yz^2 = 2x^2y$. Hence $z^2 = 2y^2$ and $z^2 = 2x^2$.

~~This~~ This gives $x^2 = y^2$.

From $g = 0$, we have $x^2 + y^2 + z^2 = 1 \Rightarrow x^2 + 3y^2 = 1$

so $y = \pm \frac{1}{\sqrt{2}}$. This gives rise to a bunch of points:

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}} \right), \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right),$$

$$\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}} \right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}} \right), \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \right)$$

and $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}} \right)$.