Math 307
Fall 2018
Exam 1 - Practice
9/20/18
Time Limit: 75 Minutes

Name (Print): KEY

Problem	Points	Score
1	20	
2	30	
3	25	
4	20	
5	30	
6	20	
7	20	
Total:	165	

1. (20 points) Which of the following matrices are in reduced row-echelon form?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

yes yes no yes yes

2. (a) (5 points) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(b) (5 points) $\begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{pmatrix} 8 & 4 \\ -1 & -1 \end{pmatrix}$

(c) (5 points) If A is a 4×3 and B is a 3×4 matrix, what are the dimensions of AB and BA?

4 x 4

(d) (5 points) True of false: $\det(AB) = \det(A)\det(B)$ True!

(e) (5 points) True of false: $\det(A + B) = \det(A) + \det(B)$ False!

(f) (5 points) Prove or provide a counter example to the following statement: If A is invertible and B is invertible then $A + B$ is invertible.

~~Consider~~ Counter example:

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Both invertible but $A + B = 0$ and 0 is not invertible

3. (a) (15 points) Prove that A and B are row equivalent if and only if there is an invertible matrix C such that $CA = B$.

($\Rightarrow$) If A and B are row equivalent, then there are elementary matrices $E_1, \dots, E_n$ such that $E_n \cdots E_1 A = B$. Because each E_i is invertible, $E_n \cdots E_1$ is too.

($\Leftarrow$) Suppose there is an invertible C such that $CA = B$. As C is invertible, it is row equivalent to the identity matrix, and, is therefore equal to a product of elementary matrices, $C = E_k \cdots E_1$. Now $E_k \cdots E_1 A = B$ tells us that A and B are row equivalent.

(b) (10 points) Compute

$$\det\left(\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 3 & 4 & 1 \\ -1 & -3 & -4 & -1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 3 & 4 & 1 \\ -1 & -3 & -4 & -1 \end{bmatrix}_{R_4 + R_3}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= 0$$

4. (a) (10 points) Compute

$$\det\left(\begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 4 & 0 \end{bmatrix}\right)$$

Expanding about the 2nd column

$$= 1\begin{vmatrix} -1 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 4 & 0 \end{vmatrix}$$

$$= 1\cdot(-4)\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= -4(-1-2)$$

$$= 12$$

(b) (10 points) Suppose A is a 5×5 matrix such that $\det(A) = 2$. Let $E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

What is $\det(EA) =$?

Because EA performs a row switch on A,

$$\det(EA) = -\det(A)$$

$$= -2$$

5. (a) (10 points) Show that $x^2+1, x-1$ are linearly independent.

If $c_1(x^2+1)+c_2(x-1)=0$

then $c_1=0$, $c_2=0$ and $c_1-c_2=0$.

This tells you that they are linearly independent.

(b) (10 points) Show that the vectors $x^2+1, x-1$ do not span P_2.

One needs, at minimum, three vectors to span P_2 since $\dim(P_2)=3$.

(c) (10 points) Show that x^2+x+1, x^2-1, $x+5$, 4 are linearly **dependent**.

Again, $\dim(P_2)=3$, and there are 4 vectors.

6. (20 points) Suppose that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine if A is invertible by computing $\det(A)$. If A *is* invertible, find A^{-1}.

By inspection, we see that A is a product of elementary matrices, and thus, as each matrix is invertible, A is too.

$$\det(A) = \det(E_4 \cdots E_1)$$

$$= \det(E_4)\det(E_3)\cdots\det(E_1)$$

$$= 2 \cdot 1 \cdot 1 \cdot (-1)$$

$$= -2$$

$$E_4^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad A = E_4 \cdots E_1 \Rightarrow A^{-1} = E_1^{-1} \cdots E_4^{-1}$$

$$A^{-1} = E_1^{-1} E_2^{-1} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = E_1^{-1} \begin{pmatrix} \frac{1}{2} & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ \frac{1}{2} & -2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

7. (20 points) Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

Find a basis for the null space and row space of A, the dimension of the null space of A, and the rank of A.

If $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, Then we have the following equations,

$$x_1 = x_3 - x_5 - 3x_7$$
$$x_2 = -x_5 - x_7$$
$$x_4 = -4x_5 - 2x_7$$
$$x_6 = -3x_7$$

This says that vectors of the form

$$\begin{pmatrix} x_3 - x_5 - 3x_7 \\ -x_5 - x_7 \\ x_3 \\ -4x_5 - 2x_7 \\ x_5 \\ -3x_7 \\ x_7 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -1 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} -3 \\ -1 \\ 0 \\ -2 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

$\uparrow$ v_1 $\uparrow$ v_2 $\uparrow$ v_3

span the null space. Also, they are linearly independent, so v_1, v_2, v_3 form a basis for $NS(A)$. ~~Also, as each w_i is linearly independent~~

The vectors ~~aaa~~ w_1, w_2, w_3 and w_4 are linearly independent. Since A is a 4×7 matrix, $\dim(NS(A)) + \dim(RS(A)) = 7 \Rightarrow \dim(RS(A)) = 4$, So, the w_i form a basis for $RS(A)$. The rank of A is 4.