Math 307
Fall 2018
Exam 2
10/25/18
Time Limit: 75 Minutes

Name (Print): Solutions

Problem	Points	Score
1	15	
2	20	
3	35	
4	30	
5	20	
6	20	
7	20	
8	20	
Total:	180	

1. Let $S : \mathbb{R}^2 \to R^2$ and $T : R^2 \to \mathbb{R}^2$ be transformations defined by

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ x-y \end{bmatrix}, \qquad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-4y \\ 3x \end{bmatrix}$$

(a) (5 points) Show that S and T are both linear transformations.

S:

① $S\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = S\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1+x_2)+y_1+y_2 \\ x_1+x_2-(y_1+y_2) \end{bmatrix} = \begin{bmatrix} 2x_1+y_1 \\ x_1-y_1 \end{bmatrix} + \begin{bmatrix} 2x_2+y_2 \\ x_2-y_2 \end{bmatrix}$

② $S\left(c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = S\left(\begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}\right) = \begin{bmatrix} 2cx_1+cy_1 \\ cx_1-cy_1 \end{bmatrix} = c\begin{bmatrix} 2x_1+y_1 \\ x_1-y_1 \end{bmatrix} = cS\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$= S\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + S\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

① + ② show that S is linear.

T:

① $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2-4(y_1+y_2) \\ 3(x_1+x_2) \end{bmatrix} = \begin{bmatrix} x_1-4y_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} x_2-4y_2 \\ 3x_2 \end{bmatrix}$

② $T\left(c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}\right) = \begin{bmatrix} cx_1-4cy_1 \\ 3cx_1 \end{bmatrix} = c\begin{bmatrix} x_1-4y_1 \\ 3x_1 \end{bmatrix} = cT\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$= T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

(b) (5 points) Find $ST\begin{bmatrix} x \\ y \end{bmatrix}$ and $T^2\begin{bmatrix} x \\ y \end{bmatrix}$.

$ST\begin{pmatrix} x \\ y \end{pmatrix} = S\begin{pmatrix} x-4y \\ 3x \end{pmatrix} = \begin{bmatrix} 2(x-4y)+3x \\ x-4y-3x \end{bmatrix}$

$T^2\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{pmatrix} x-4y \\ 3x \end{pmatrix} = \begin{bmatrix} x-4y-4(3x) \\ 9x \end{bmatrix}$

Sorry this 9x should be a 3(x-4y)

(c) (5 points) Find the matrices of S and T with respect to the standard basis for $\mathbb{R}^2$.

$S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If $\alpha = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$, then $[S]_\alpha^\alpha = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$.

(BTW: You can check this by computing $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = S\begin{pmatrix} x \\ y \end{pmatrix}$)

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} = -4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so, $[T]_\alpha^\alpha = \begin{pmatrix} 1 & -4 \\ 3 & 0 \end{pmatrix}$. (Again, you can check this)

2. (a) (10 points) Prove or provide a counter example: If T is a linear transformation, then so is $T + T^2$.

This is a real life fact:

$$(T+T^2)(x+y) = T(x+y) + T^2(x+y)$$
$$= T(x)+T(y) + T(T(x)+T(y))$$
$$= T(x) + T(T(x)) + T(y) + T(T(y))$$
$$= (T+T^2)(x) + (T+T^2)(y)$$

$$(T+T^2)(cx) = T(cx) + T^2(cx)$$
$$= cT(x) + T(T(cx))$$
$$= cT(x) + cT(T(x)) = c(T+T^2)(x)$$

(b) (10 points) Let $D : C^\infty(-\infty, \infty) \to C^\infty(-\infty, \infty)$ be the usual derivative operator. Find a basis for the kernel of the operator

$$D^2 - 4D + 4$$

The characteristic polynomial here is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, which tells us that $y = e^{2x}$ and $y = xe^{2x}$ are the only solutions to $y'' - 4y' + 4y = 0$. Since solutions to this differential equation are elements of the kernel of $D^2 - 4D + 4$ we only need to show that they are linearly independent (as this is a condition needed to be a basis):

$$W = \begin{pmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{pmatrix}$$ and so $\det(W) = (e^{4x} + 2xe^{4x}) - 2xe^{4x}$
$$= e^{4x} \neq 0.$$

So, $f(x) = e^{2x}$ and $g(x) = xe^{2x}$ form a basis for the kernel of said operator.

3. Let α be the standard basis for $\mathbb{R}^3$ and β the basis consisting of

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

(a) (5 points) Find the change of basis matrix from α to β.

This is the matrix $[I]_\beta^\alpha$:

$$I\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$I\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$I\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&1&0\\0&1&0\\1&0&1\end{pmatrix}$$

(b) (10 points) Find the change of basis matrix from β to α.

Recall that one may find $\left([I]_\beta^\alpha\right)^{-1} = [I]_\alpha^\beta$ = The change of basis matrix from β to α.

$$\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&0&0&1&0\\1&0&1&0&0&1\end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&0&1&0\\1&0&1&0&0&1\end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc}1&0&0&1&-1&0\\0&1&0&0&1&0\\0&0&1&-1&1&1\end{array}\right) \therefore [I]_\alpha^\beta = \begin{pmatrix}1&-1&0\\0&1&0\\-1&1&1\end{pmatrix}$$

(c) (10 points) Define $T : \mathbb{R}^3 \to R^3$ by $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\2y\\3z\end{bmatrix}$, Find $[T]_\alpha^\alpha$.

$$[T]_\alpha^\alpha = \begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$$

(d) (10 points) Express $[T]_\beta^\beta$ as the product of the three matrices found above.

Since $[T]_\beta^\beta = [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha$, we have that

$$[T]_\beta^\beta = \begin{pmatrix}1&-1&0\\0&1&0\\-1&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\1&0&1\end{pmatrix}$$

4. (a) (10 points) Let A be an $n \times n$ matrix. Prove that a number λ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0.$$

($\Rightarrow$) Suppose that λ is an eigenvalue of A. Then, there is a non-zero v such that $Av = \lambda v$. This gives us that $0 = \lambda v - Av = (\lambda I - A)v$.

Since $v \neq 0$, this says that $\lambda I - A$ is not invertible and thus, $\det(\lambda I - A) = 0$.

($\Leftarrow$) If $\det(\lambda I - A) = 0$ then $(\lambda I - A)$ is not invertible and whence there is a non-zero v such that $0 = (\lambda I - A)v$, so $Av = \lambda v$ (so λ is an eigenvalue of A).

(b) (10 points) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalue(s) and associated eigenvectors of A.

$\lambda I - A = \begin{pmatrix} \lambda - 2 & -2 \\ -1 & \lambda - 3 \end{pmatrix}$, and $\det(\lambda I - A) = (\lambda - 2)(\lambda - 3) - 2$

$= \lambda^2 - 5\lambda + 6 - 2$

$= \lambda^2 - 5\lambda + 4$

$= (\lambda - 1)(\lambda - 4)$.

$\lambda = 1$: $\begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

This matrix has null-space w/ basis vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\therefore E_1 = \text{span}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda = 4$: $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, This matrix has null space w/ basis vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore E_4 = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (One can check that $A\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.)

(c) (10 points) Define what it means for A to be similar to a matrix B, and what it means for A to be diagonalizable. Prove that if A is similar to B, and A is diagonalizable, then B is also diagonalizable.

A is similar to B is $A = P^{-1}BP$ for some invertible P.

A is diagonalizable if A is similar to a diagonal matrix.

If $A = P^{-1}BP$ and $A = Q^{-1}DQ$ for invertible P, Q and diagonal D, then

$$Q^{-1}DQ = P^{-1}BP$$

and thus $B = PQ^{-1}DQP^{-1}$

$= (QP^{-1})^{-1} D (QP^{-1})$.

Hence, B is diagonalizable.

5. Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) (10 points) Find the eigenvalues and associated eigenvectors of A.

$\det(\lambda I - A) = (\lambda-2)(\lambda-3)(\lambda-1)$

$\lambda=2$: $\begin{pmatrix} 0 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$ eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda=3$: $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$ eigenvector $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$\lambda=1$: $\begin{pmatrix} -1 & -2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$ eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(b) (10 points) Determine if A is diagonalizable. If it is, give the matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

A is 3×3 and $\dim(E_2) + \dim(E_3) + \dim(E_1) = 3$,

so, yes, A is diagonalizable.

If $P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then $P^{-1}AP = D$.

6. Suppose that A is a matrix with characteristic polynomial $p(\lambda) = (\lambda - 3)^2(\lambda - 2)^2$.

(a) (10 points) If $\dim(E_3) = 2$ and $\dim(E_2) = 2$ what is the Jordan Normal Form of A?

Since $\dim(E_3) + \dim(E_2) \overset{a?}{=} 2 + 2 = 4$, A must be diagonalizable, so,

$$J_A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(b) (10 points) If $\dim(E_3) = 1$ and $\dim(E_2) = 2$ what is the Jordan Normal Form of A?

$$J_A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

7. (20 points) Suppose that A is a matrix with characteristic polynomial $p(\lambda) = (\lambda+1)^2(\lambda-5)^4$. If we decide on a Jordan Normal Form, J, of A as

$$J = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where B_1 is a 2×2 matrix and B_2 is a 4×4 matrix, what are the possibilities (up to permutation of the Jordan blocks) of B_1 and B_2?

B_1 could be $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

B_2 could be

$\begin{pmatrix} 5 & 0 & 0 & \\ & 5 & 0 & \\ & 0 & 5 & \\ & & & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

$\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ or $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

(Sorry this got messy)

8. (20 points) Is the following $n \times n$ matrix diagonalizable?

$$\begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The only eigenvalue is $\lambda = 1$.

$$1I - A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The nullspace of this matrix has dimension $n-1$,

so, no, this matrix is not diagonalizable.