Solutions by Curlee

Math 307
Fall 2018
Final - Practice
12/4/18
Time Limit: 120 Minutes

Name (Print): _____

Problem	Points	Score
1	20	
2	15	
3	15	
4	20	
5	50	
6	25	
Total:	145	

1. (20 points) Answer true or false to the following statements:

If V is a vector space of dimension n, then V is an $n \times n$ matrix. F

If 0 is an eigenvalue of A, then A is not invertible. T

If V is a vector space of dimension n, then any set of n vectors must be linearly independent. F

If $\det(A) \neq 0$ then A is invertible. T

If A is an $n \times n$ invertible matrix, then the rows of A are linerarly independent T

If v_1 and v_2 are linearly independent then there exists nonzero constants, c_1 and c_2, such that $c_1v_1 + c_2v_2 = 0$. F

If λ is an eigenvalue of A then $\det(\lambda I - A) = 0$. T

If V is a vector space of dimension n, then any collection of n vectors forms a basis for V. F

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable. T

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is invertible. F

If A is an $n \times n$ matrix that is invertible, then A is diagonalizable. F

If A is an $n \times n$ matrix that is diagonalizable, then A is invertible. F

2. (15 points) Consider the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Determine $\dim(E_1)$, $\dim(E_2)$ and $\dim(E_3)$.

$\dim(E_1) = 2$, $\dim(E_2) = 1$, $\dim(E_3) = 1$

3. Consider the homogeneous first order linear differential system

$$\begin{aligned} x' &= x - y + z \\ y' &= 2y + z \\ z' &= 2z \end{aligned}$$

where x, y, z are all functions of the variable t.

(a) (5 points) Write the system in the matrix form $Y' = AY + G$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 0$$

(b) (5 points) Is the matrix A diagonalizable? Explain your answer.

One can check that $\dim(E_2) = 1$ and $\dim(E_1) = 1$ and therefore A is not diagonalizable.

(c) (5 points) Find the general solution to the homogeneous system $Y' = AY$.

As $z' = 2z$ we have $\boxed{z = c_3 e^{2t}}$. Now, $y' = 2y + c_3 e^{2t}$.

$$e^{-2t}(y' - 2y) = c_3 e^{2t} \cdot e^{-2t}$$

$$\frac{d}{dt}(y e^{-2t}) = c_3$$

$$\boxed{y = c_3 t e^{2t} + c_2 e^{2t}}$$

Farther,

$$x' = x - (c_3 t e^{2t} + c_2 e^{2t}) + c_3 e^{2t}$$

$$e^{-t}(x' - x) = ((c_3 - c_2) e^{2t} - c_3 t e^{2t}) e^{-t}$$

$$\frac{d}{dt}(x e^{-t}) = (c_3 - c_2) e^{t} - c_3 t e^{t}$$

$$x e^{-t} = (c_3 - c_2) e^{t} - c_3 (t e^{t} - e^{t}) + c_1$$

$$\therefore \boxed{x = (c_3 - c_2) e^{2t} - c_3 (t e^{2t} - e^{2t}) + c_1 e^{t}}$$

4. (20 points) Solve the initial value problem

$$Y' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} Y + \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = A$

First we solve the homogenous case, $\Psi' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \Psi$.

The eigenvalues of $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ are $2 \pm i$. If $P^{-1}AP = \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$ then $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Notice that $\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{(2+i)x} \\ 0 \end{pmatrix} = \begin{pmatrix} ie^{(2+i)x} \\ e^{(2+i)x} \end{pmatrix}$

and that $\begin{pmatrix} ie^{(2+i)x} \\ e^{(2+i)x} \end{pmatrix} = \begin{pmatrix} -e^{2x}\sin(x) \\ e^{2x}\cos(x) \end{pmatrix} + i\begin{pmatrix} e^{2x}\cos(x) \\ e^{2x}\sin(x) \end{pmatrix}$.

Thus our matrix of fundamental solutions is

$M = \begin{pmatrix} -e^{2x}\sin(x) & e^{2x}\cos(x) \\ e^{2x}\cos(x) & e^{2x}\sin(x) \end{pmatrix}$, and one can show that

$M^{-1} = \begin{pmatrix} -e^{-2x}\sin(x) & e^{-2x}\cos(x) \\ e^{-2x}\cos(x) & e^{-2x}\sin(x) \end{pmatrix}$. Now one finds

the particular solution by computing

$$\Psi_P = M \int M^{-1} G, \quad (\leftarrow \text{doing this by hand is possible ....})$$

and our general solution is

$$\Psi_H + \Psi_P$$

where $\Psi_H = c_1\begin{pmatrix} -e^{2x}\sin(x) \\ e^{2x}\cos(x) \end{pmatrix} + c_2\begin{pmatrix} e^{2x}\cos(x) \\ e^{2x}\sin(x) \end{pmatrix}$.

From here, solving for c_1 and c_2 is easy.

5. Let's consider the non-homogeneous first order linear differential system

$$\begin{aligned} x' &= -4x - 3y + 3z \\ y' &= 3x + 2y - 3z + e^t \\ z' &= -3x - 3y + 2z \end{aligned}$$

where x, y, z are all functions of the variable t.

(a) (5 points) Write the system in the matrix form $Y' = AY + G$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -4 & -3 & 3 \\ 3 & 2 & -3 \\ -3 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

(b) (15 points) Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

One finds the eigenvalues of A to be 2 and -1, with the multiplicity of $\lambda = 2$: one, and $\lambda = -1$: two.

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(c) (5 points) Find the general solution to the homogeneous system $Z' = DZ$.

If $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, then $z_1 = c_1 e^{2x}$, $z_2 = c_2 e^{-x}$ and $z_3 = c_3 e^{-x}$.

(d) (5 points) Use the result of the previous question to find the general solution Y_H to the homogeneous system $Y' = AY$, and find a fundamental set, Y_1, Y_2, Y_3, of solutions to this system.

Here If $\varphi = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ then our solutions are

$$\varphi_H = PZ = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 e^{2x} \\ c_2 e^{-x} \\ c_3 e^{-x} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{2x} + c_2 e^{-x} - c_3 e^{-x} \\ -c_1 e^{2x} + c_3 e^{-x} \\ c_1 e^{2x} + c_2 e^{-x} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} e^{2x} \\ -e^{2x} \\ e^{2x} \end{pmatrix} + c_2 \begin{pmatrix} e^{-x} \\ 0 \\ e^{-x} \end{pmatrix} + c_3 \begin{pmatrix} -e^{-x} \\ e^{-x} \\ 0 \end{pmatrix}$$

φ_1 φ_2 φ_3

(e) (15 points) Using the matrix $M = [Y_1\ Y_2\ Y_3]$, compute a particular solution Y_p to the non-homogeneous system $Y' = AY + G$.

Here $M = \begin{pmatrix} e^{2x} & e^{-x} & -e^{-x} \\ -e^{2x} & 0 & e^{-x} \\ e^{2x} & e^{-x} & 0 \end{pmatrix}$ and one can find $M^{-1} = \begin{pmatrix} -e^{-2x} & -e^{-2x} & e^{-2x} \\ e^{x} & e^{x} & 0 \\ -e^{-x} & 0 & e^{x} \end{pmatrix}$

Now, finding $Y_p = M\int M^{-1}G$ is not too hard:

$\int M^{-1}G = \int \begin{pmatrix} -e^{x} \\ e^{2x} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{x} \\ \frac{1}{2}e^{2x} \\ 0 \end{pmatrix}$ and thus, $M\int M^{-1}G = \begin{pmatrix} \frac{1}{2}e^{x} \\ e^{x} \\ \frac{3}{2}e^{x} \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{2}e^{x} \\ -e^{x} \\ \frac{3}{2}e^{x} \end{pmatrix}$

(f) (5 points) Use the result of the previous question to find the general solution to the non-homogeneous system $Y' = AY + G$.

General Solution:

$$C_1\begin{pmatrix} e^{2x} \\ -e^{2x} \\ e^{2x} \end{pmatrix} + C_2\begin{pmatrix} e^{-x} \\ 0 \\ e^{-x} \end{pmatrix} + C_3\begin{pmatrix} -e^{-x} \\ e^{-x} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{3}{2}e^{x} \\ -e^{x} \\ \frac{3}{2}e^{x} \end{pmatrix}$$

6. Consider the following system:

$$y_1'' = y_1' + y_2' + e^x$$
$$y_2'' = y_1 + y_2 + \sin(x)$$

where y_1 and y_2 are functions of x.

(a) (5 points) Write this system as a system of first order liner differential equations.

Set $V_1 = y_1$, $V_2 = y_1'$, $V_3 = y_2$, $V_4 = y_2'$

Now, $V_1' = V_2$

$V_2' = V_2 + V_3 + e^x$

$V_3' = V_3$

$V_4' = V_1 + V_3 + \sin(x)$

(b) (5 points) Write the following differential equation as a system of first order linear equations:

$$y'' + 4y' + 3y = 0$$

$V_1 = y$, $V_2 = y'$ $\Rightarrow$ $V_1' = V_2$, $V_2' = -4V_2 - 3V_1$

aka $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}}_{A} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

(c) (10 points) Solve the system from part b).

A from part (b) has eigenvalues -1 and -3 with eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Since $\begin{pmatrix} e^{-x} \\ e^{-3x} \end{pmatrix}$ solves $Z' = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} Z$,

our general solution is $\begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{-x} \\ c_2 e^{-3x} \end{pmatrix}$

$= c_1 \begin{pmatrix} -e^{-x} \\ e^{-x} \end{pmatrix} + c_2 \begin{pmatrix} -e^{-3x} \\ 3e^{-3x} \end{pmatrix}$

(d) (5 points) Verify your solution by solving the original differential equation by using it's characteristic polynomial.

(from (c), we find that $y = V_1 = -c_1 e^{-x} - c_2 e^{-3x}$)

$\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) \Rightarrow y = \bar{c}_1 e^{-x} + \bar{c}_2 e^{-3x}$.