MATH 241, Fall '11
Final Exam

Name: Solutions by
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INSTRUCTIONS: Write legibly. Indicate your answer clearly. Show all work; explain your answers. Answers with work not shown might be worth **zero** points. No calculators, cell phones, or cheating.

Problem	Worth	Score
1	24	
2	24	
3	10	
4	28	
5	12	
6	16	
7	16	
8	8	
9	16	
10	18	
11	16	
12	12	
Total	200	

(6) 0. **Extra Credit:** Show that for all x and y, $|\sin(x) - \sin(y)| \leq |x - y|$.

For x, y in $\mathbb{R}$, the m.v.t. gives an α such that

$$\frac{\sin(x) - \sin(y)}{x - y} = \cos(\alpha).$$

$$\Rightarrow \quad |\sin(x) - \sin(y)| = |\cos(\alpha)|\,|x - y| \leq |x - y|$$

$\square$.

(24) 1. Compute each of the following limits or show that they do not exist. **Show your work!**

(a) $\lim_{x\to -3} \frac{x+3}{x^2+7x+12}$ $= \lim_{x\to -3} \frac{x+3}{(x+3)(x+4)} = \lim_{x\to -3} \frac{1}{x+4}$

$= 1$

(b) $\lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}$ $= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$

$= \frac{1}{6}$

(c) $\lim_{x\to 0} \frac{x}{\sin 3x}$ $= \lim_{x\to 0} \frac{1}{3} \frac{3x}{\sin(3x)} = \frac{1}{3}\cdot 1 = 1/3$

(d) $\lim_{x\to 2^+} \frac{|x-2|}{2-x}$

$x-2 \geq 0$ when $x \geq 2$

$\Rightarrow |x-2| = x-2$ as $x \to 2^+$.

$\lim_{x\to 2^+} \frac{x-2}{2-x} = \lim_{x\to 2^+} -1 = -1$

(24) 2. Find the derivatives of each of the following functions. **Do not simplify!**

(a) $f(x) = x^2\sqrt{\sin x}$

$$f'(x) = 2x\sqrt{\sin(x)} + \frac{1}{2\sqrt{\sin(x)}} \cdot \cos(x) \cdot x^2$$

(b) $g(x) = \dfrac{x^2+x+1}{\sqrt{x^2+1}}$

$$g'(x) = \frac{(2x+1)\sqrt{x^2+1} - \frac{1}{2\sqrt{x^2+1}} \cdot 2x\,(x^2+x+1)}{x^2+1}$$

(c) $h(x) = \displaystyle\int_{\sqrt{x}}^{5} \sin^5 t\, dt$

$$h'(x) = -\sin^5(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

(10) 3. Let $f(x) = \dfrac{1}{3x}$. Use the *definition of the derivative* to compute $f'(2)$. **No work, no credit.**

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{3(2+h)} - \frac{1}{6}}{h}$$

$$= \lim_{h\to 0} \frac{6 - 3(2+h)}{18(2+h)h}$$

$$= \lim_{h\to 0} \frac{-3}{18(2+h)} = \frac{-3}{18\cdot 2}$$

$$= \frac{-1}{12}$$

$\cancel{\frac{-3}{(3x)^2}}$

$\cancel{\frac{-3}{6^2} = \frac{-3}{36} = \frac{-1}{12}}$

(28) 4. Evaluate the following integrals. **Show your work!**

(a) $\int x^2\sqrt{1+10x^3}\,dx$ $u = 1+10x^3$, $du = 30x^2dx$

$\Rightarrow \frac{1}{30}du = x^2dx$

$\frac{1}{30}\int\sqrt{u}\,du$

$= \frac{1}{30}\frac{u^{3/2}}{3/2} + C = \frac{2}{90}u^{3/2}+C$

$= 2/90(1+10x^2)^{3/2} + C$

(b) $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$ $u = x+1$, $du = dx$

$u-1 = x$

$\int_1^2 \frac{u-1}{\sqrt{u}}\,du = \int_1^2 \sqrt{u} - u^{-1/2}\,du$

$= \frac{u^{3/2}}{3/2} - 2u^{1/2}\Big|_1^2 = \frac{2}{3}(2)^{3/2} - 2\,2^{1/2}$

$-\left(\frac{2}{3}-2\right)$

(c) $\int_0^1 (x^2+1)(3x-2)\,dx$

$= \int_0^1 3x^3+3x-2x^2-2\,dx$

$= \frac{3x^4}{4} - \frac{3x^2}{2} - \frac{2x^3}{3} - 2x\Big|_0^1$

$= \frac{3}{4} - \frac{3}{2} - \frac{2}{3} - 2$

(d) $\int_0^{\pi/2} \sin x\cos^3 x\,dx$

$u = \cos(x)$, $du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$-\int_1^0 u^3du = -\left(u^4/4\Big|_{u=1}^{u=0}\right)$

$= -\left(0-\frac{1}{4}\right) = \frac{1}{4}$

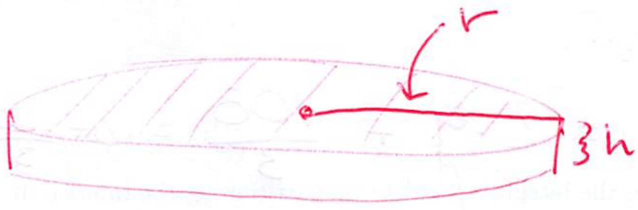
(12) 5. Find the equation of the tangent line to the curve $x^3 + y^3 = 9xy$ at the point $(4, 2)$. **Show your work!**

$$\Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 9\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y-3x^2}{3y^2-9x}, \quad \left.\frac{dy}{dx}\right|_{(4,2)} = \frac{5}{4}$$

(16) 6. 100m^3 of oil is spilled when a tanker collides with a tuna boat. The resulting oil slick forms a right circular cylinder on the surface of the water. If the thickness (h) of the slick is decreasing at a rate of 0.001 m/sec, how fast is the radius (r) increasing when the slick is 0.01 m thick? Note: $V = \pi r^2 h$.



Given: $\frac{dV}{dt} = 0$

$\frac{dh}{dt} = -.001$ m/sec.

$$\frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h)$$

Note: when $h = .01$,

$100 = \pi (r)^2 .01$

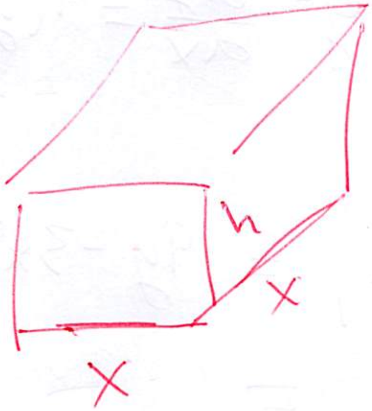
$\Rightarrow r = \sqrt{\frac{100 \cdot 100}{\pi}} = \frac{100}{\pi}$

$$\Rightarrow 0 = \pi\left(2r \cdot \frac{dr}{dt} h + r^2 \frac{dh}{dt}\right)$$

$$0 = \pi\left(2\frac{100}{\pi} \cdot \frac{dr}{dt} \cdot (.01) + \left(\frac{100}{\pi}\right)^2 \cdot (-.001)\right)$$

Now, solve for dr/dt!

(16) 7. A rectangular box with volume 18 ft^3 is to be built with a square base and no top. The material used for the bottom panel costs \$ 2.00 per ft^2 while the material used for the side panels cost \$ 1.50 per ft^2. Find the minimum cost of such a box. **Justify your answer using the methods of calculus.**



Cost: $2 \cdot x^2 + 1.5(4xh)$

$= 2x^2 + 6xh$

$V = x^2 h \Rightarrow h = \frac{18}{x^2}$

$\therefore C = 2x^2 + 6x\left(\frac{18}{x^2}\right)$

$= 2x^2 + \frac{108}{x}$

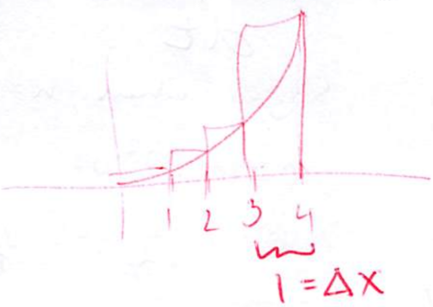
$C' = 4x - \frac{108}{x^2} = \frac{4x^3 - 108}{x^2}$

$C' = 0$ when $4x^3 - 108 = 0 \Rightarrow 4(x^3 - 27) = 0 \Rightarrow \underline{x = 3}$

This is a min (why?).

Now, $C(3) = 2 \cdot 9 + \frac{108}{3} = 18 + \frac{108}{3}$ is max. vol.

(8) 8. Set up the Riemann sum approximation to the integral $\int_0^4 x^3\,dx$ by partitioning the interval $[0, 4]$ into 4 subintervals of equal length and using the right endpoint of each subinterval to calculate the height of the corresponding rectangle.



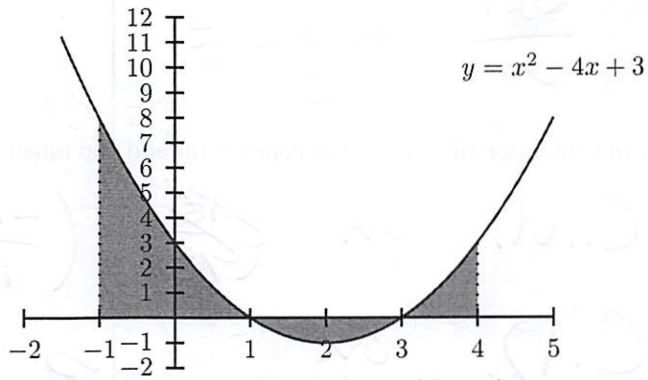
Approx: $\underline{1^3 + 2^3 + 3^3 + 4^3}$

(16) 9. Let $f(x) = x^2 - 4x + 3$. The graph of $y = f(x)$ is shown below.

(a) Compute $\int_{-1}^{4} f(x)\,dx$ $= \frac{x^3}{3} - 2x^2 + 3x \Big|_{-1}^{4}$

$$= \frac{4^3}{3} - 2(4)^2 + 3(4) - \left(\frac{(-1)^3}{3} - 2(-1)^2 + 3(-1)\right)$$

(b) Find the total area between the graph and the x–axis for x between -1 and 4.



Graph for problem above

$$\int_{-1}^{1} x^2 - 4x + 3\,dx + \int_{1}^{3} -(x^2 - 4x + 3)\,dx + \int_{3}^{4} x^2 - 4x + 3\,dx$$

$$\frac{x^3}{3} - 2x^2 + 3x \Big|_{-1}^{1} + -\frac{x^3}{3} + 2x^2 - 3x \Big|_{1}^{3} + \frac{x^3}{3} - 2x^2 + 3x \Big|_{3}^{4}$$

Yeah, no.

(18) 10. Let $f(x) = \frac{x^2}{(x-3)^2}$. Answer the question below. **Show all reasoning using the methods of calculus.** Note: $f'(x) = \frac{-6x}{(x-3)^3}$ and $f''(x) = \frac{12x+18}{(x-3)^4}$.

(a) Find all points where f is not continuous.

$x = 3$. It's the only point f is not defined.

(b) Find the intervals where f is increasing and the intervals where f is decreasing.

Decreasing on $(0, \infty)$, Increasing on $(-\infty, 0)$.

(c) Find the intervals where f is concave up and the intervals where f is concave down.

C.U. on $\cancel{\frac{18}{18}}$ $\left(\frac{-18}{12}, \infty\right)$

C.D. on $\left(-\infty, \frac{-18}{12}\right)$

(d) Find all local extrema.

One, @ $x=0$.

(e) Find all inflection points.

One, @ $x = \frac{-18}{12}$

(f) Find the equations of all asymptotes.

$\lim_{x \to \infty} \frac{x^2}{(x-3)^2} = 1$ (why?)

$\Rightarrow$ H.A. of $y=1$

(16) 11. Solve the initial value problem (In other words, find a function $y(x)$ that satisfies both equations.):

$$\frac{dy}{dx} = 3\sin(2x) + 6 \quad \text{and} \quad y(0) = 1.$$

$$y = \frac{-3\cos(2x)}{2} + 6x + C$$

$$1 = \frac{-3}{2}\cos(2(0)) + 6(0) + C$$

$$\Rightarrow C = 5/2, \quad \therefore y = \frac{-3}{2}\cos(2x) + 6x + \frac{5}{2}.$$

(12) 12. Let $f(x) = 3x^2 + 5x - 9$.

(a) Explain why f satisfies the hypotheses of the Mean Value Theorem over the interval $[0, 3]$.

It's a polynomial, and $\therefore$ cont. and diff. everywhere.

(b) Find a point $c \in (0, 3)$ such that the slope of the tangent line at $(c, f(c))$ is the average rate of change of f over the interval.

Hint: find a c such that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} !$$