

Fall '17

MATH 241/251A FINAL EXAM

Your name: Solutions
by Carter

Select your instructor and section time:

- Luca Candelori (Thursday 1:30pm)
- Luca Candelori (Friday 10:30am)
- Erik Guentner (Wednesday 8:30am)
- Asaf Hadari (Thursday 10:30am)
- Piper Harron (Thursday 12:00pm)
- Piper Harron (Friday 9:30am)
- Mushfeq Khan (Wednesday 10:30)
- Mushfeq Khan (Wednesday 1:30pm)
- Daisuke Takagi (Thursday 1:30pm)
- Daisuke Takagi (Friday 11:30am)
- David Webb (Friday 8:30)
- David Yuen (Thursday 8:30am)
- David Yuen (Thursday 10:30am)

1 (16)	
2 (4)	
3 (10)	
4 (15)	
5 (3)	
6 (10)	
7 (12)	
8 (10)	
9 (10)	
10 (10)	
11 (18)	
12 (6)	
13 (6)	
14 (10)	
TOTAL (140)	

Justify all your work. Answers without suitable justification will receive no credit.

Problem 1. (16 points) Evaluate the following limits. If the limit is infinite, indicate whether it is ∞ or $-\infty$. (Do not use l'Hôpital's rule.)

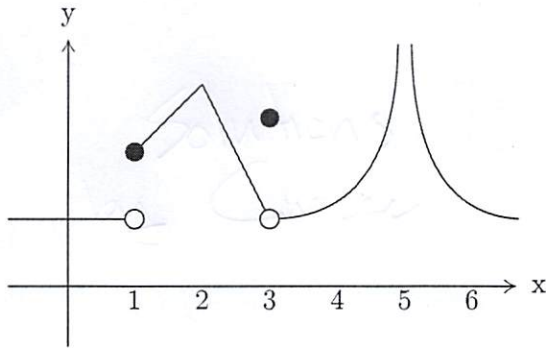
$$\begin{aligned} \text{a. } \lim_{x \rightarrow \infty} \frac{x^3 + x}{3x^3 - 1} &= \lim_{x \rightarrow \infty} \frac{x^3 \left(1 + \frac{1}{x^2}\right)}{x^3 \left(3 - \frac{1}{x^3}\right)} \\ &= \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x^2}}{3 - \frac{1}{x^3}} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{b. } \lim_{x \rightarrow 2^+} \frac{4 - 2x}{|2x - 4|} &, \text{ when } x > 2, \quad 2x - 4 > 0 \quad \text{so} \quad |2x - 4| = 2x - 4 \\ &= \lim_{x \rightarrow 2^+} \frac{4 - 2x}{2x - 4} = \lim_{x \rightarrow 2^+} \frac{-(2x - 4)}{2x - 4} = \lim_{x \rightarrow 2^+} -1 = -1 \end{aligned}$$

$$\text{c. } \lim_{x \rightarrow 0} \frac{x^2 - 4}{x - 2} = \frac{0^2 - 4}{0 - 2} = 2$$

$$\begin{aligned} \text{d. } \lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{\theta} &= \lim_{\theta \rightarrow 0} 2 \frac{\sin(2\theta)}{2\theta} \\ &= 2 \cdot 1 \\ &= 2 \end{aligned}$$

Problem 2. (4 points) Below is the graph of $y = f(x)$.



a. Find the values of a for which $\lim_{x \rightarrow a^+} f(x)$ is infinite or does not exist.

$$a = 5$$

b. Find the values of a for which $\lim_{x \rightarrow a^-} f(x)$ is infinite or does not exist.

$$a = 5$$

c. Find the values of a for which $\lim_{x \rightarrow a} f(x)$ is infinite or does not exist.

$$a = 1 \text{ and } a = 5$$

d. Find the values of a for which f is not continuous at $x = a$.

$$a = 1, a = 3, a = 5$$

Problem 3. (10 points)

a. State the definition of $f'(x)$ as a limit.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

b. Let $f(x) = \sqrt{2x}$. Use the definition of the derivative to calculate $f'(2)$ (do not use differentiation rules).

$$\begin{aligned} f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{2(2+h)} - 2}{h} \cdot \frac{\sqrt{4+2h} + 2}{\sqrt{4+2h} + 2} \\ &= \lim_{h \rightarrow 0} \frac{4+2h - 4}{h(\sqrt{4+2h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{2}{\sqrt{4+2h} + 2} \\ &= \frac{2}{2+2} \\ &= \frac{1}{2} \end{aligned}$$

Problem 4. (15 points) Find the following derivatives using differentiation rules. You do not have to simplify your answers.

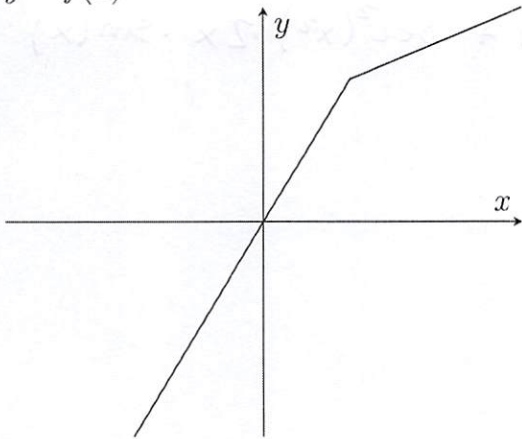
$$\text{a. } \frac{d}{dx} (\sin(x) \tan(x^2)) = \cos(x) \tan(x^2) + \sec^2(x^2) \cdot 2x \cdot \sin(x)$$

$$\text{b. } \frac{d}{dx} \left(\frac{x}{x^3 - 1} \right) = \frac{1 \cdot (x^3 - 1) - 3x^2 \cdot x}{(x^3 - 1)^2}$$

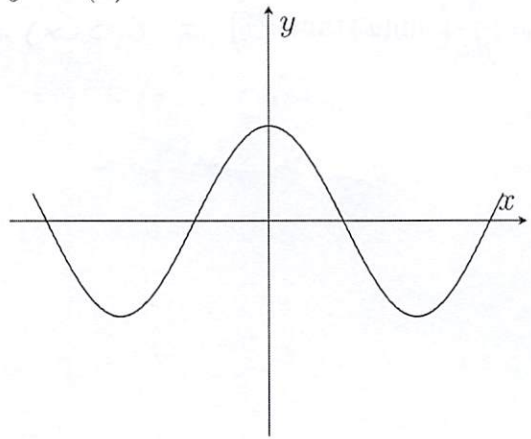
$$\text{c. } \frac{d}{dx} (\sqrt{\cos(2x + 1)}) = \frac{1}{2\sqrt{\cos(2x + 1)}} \cdot -\sin(2x + 1) \cdot 2$$

Problem 5. (3 points) Decide which function on the left has which derivative on the right.

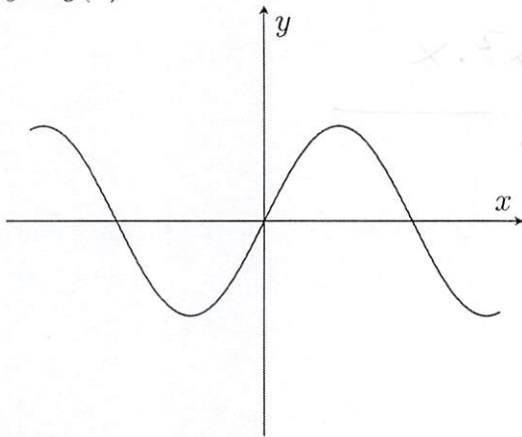
1. $y = f(x)$



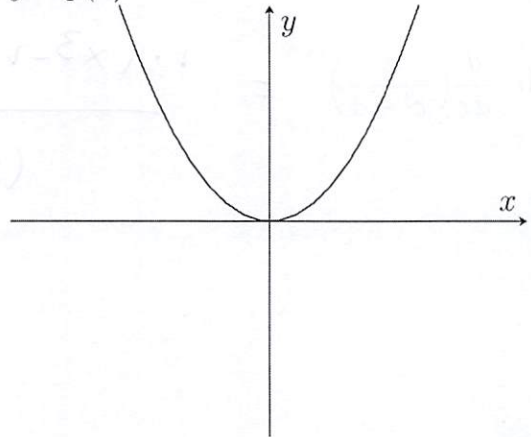
a. $y = k(x)$



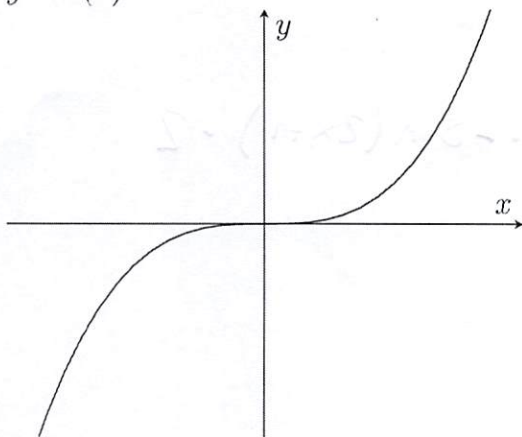
2. $y = g(x)$



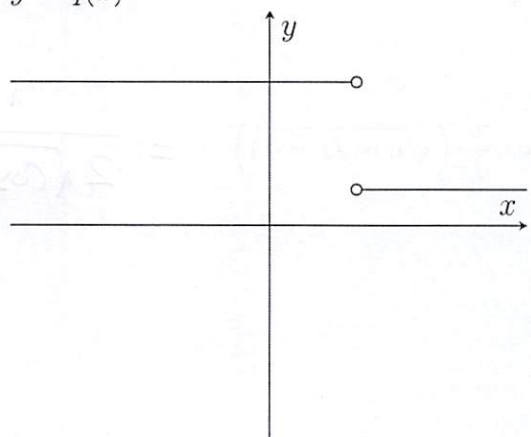
b. $y = p(x)$



3. $y = h(x)$



c. $y = q(x)$

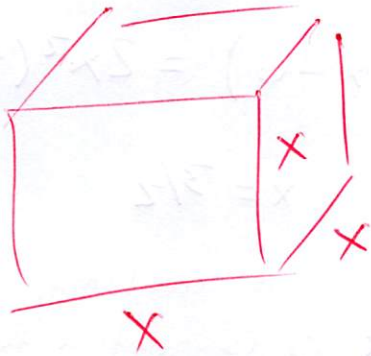


1. $f'(x) = \square k(x) \quad \square p(x) \quad \boxed{\times} q(x)$

2. $g'(x) = \boxed{\times} k(x) \quad \square p(x) \quad \square q(x)$

3. $h'(x) = \square k(x) \quad \boxed{\times} p(x) \quad \square q(x)$

Problem 6. (12 points) A cube of ice is melting evenly at a rate of $12 \text{ cm}^3/\text{hour}$. How fast is the side length of the cube changing when the side length is 4 cm ?



$$V = x^3$$

Given $\frac{dV}{dt} = -12 \text{ cm}^3/\text{H}$

when $x = 4 \text{ cm}$

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{\frac{dV}{dt}}{3x^2}$$

so, $\frac{dx}{dt} \Big|_{x=4} = \frac{-12}{3 \cdot 4 \cdot 4}$

$$= -\frac{1}{4} \text{ cm/H}$$

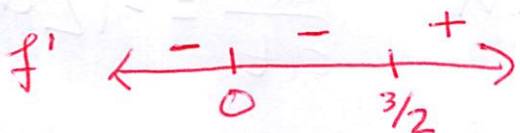
∴ Side length is shrinking at
a rate of $\frac{1}{4} \text{ cm/H}$
(Tricky huh?)

Problem 7. (12 points) Let $f(x) = x^4 - 2x^3$.

a. Find the critical points of f and classify them as local minima, local maxima or neither.

$$f'(x) = 4x^3 - 6x^2 = x^2(4x - 6) = 2x^2(2x - 3)$$

C.P.'s are $x=0$ and $x=3/2$



$x=3/2$ is a local min

and $x=0$ is not a local min or local max.

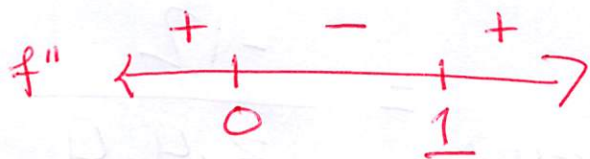
b. On which intervals is f increasing and on which is f decreasing?

$$f'(x) > 0 \text{ on } (3/2, \infty) \Rightarrow f \text{ is increasing on } (3/2, \infty)$$

$$f'(x) \leq 0 \text{ on } (-\infty, 3/2) \Rightarrow f \text{ is decreasing on } (-\infty, 3/2)$$

c. Find the inflection points of f and the intervals on which it is concave up and those on which it is concave down.

$$f''(x) = 12x^2 - 12x = 12x(x - 1)$$



C.U. on $(-\infty, 0) \cup (1, \infty)$

C.D. on $(0, 1)$

d. Find the absolute maximum and the absolute minimum of f on the interval $[-1, 1]$.

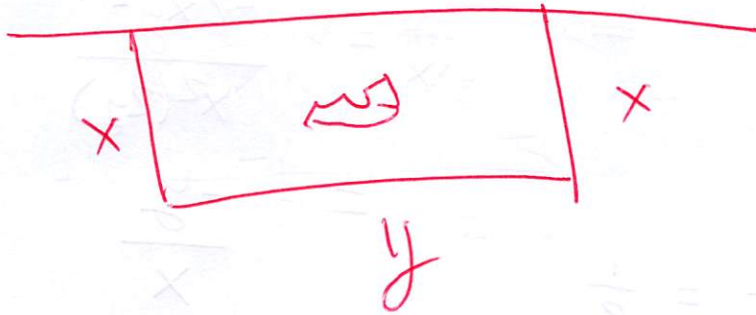
$$f(-1) = (-1)^4 - 2(-1)^3 = 1 + 2 = 3$$

$$f(0) = 0$$

$$f(1) = 1^4 - 2(1)^3 = -1$$

Abs max $\odot (-1, 3)$, Abs ~~max~~ min $\odot (1, -1)$

Problem 8. (10 points) A rectangular section of a beach reserved for monk seals is being fenced off on three sides (the fourth side borders on the ocean and does not require fencing). If there are 100m of fencing, what is the largest area that can be fenced off?



$$100 = 2x + y$$

$$A = x \cdot y = x(100 - 2x) \quad \text{on } [0, 50]$$

$$= 100x - 2x^2$$

$$A'(x) = 100 - 4x \Rightarrow (A'(x) = 0 \Rightarrow x = 25)$$

$$A' \begin{array}{c} + \\ \leftarrow \quad | \quad \rightarrow \\ 25 \end{array} \quad \infty \text{ max @ } x = 25,$$

$$A(25) = 25(50)$$

$= 1250$ is largest area.

Problem 9. (10 points) Find an equation for the tangent line to the curve $x^2y^2 = 9$ at the point $(3, -1)$.

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(9)$$

$$2xy^2 + x^2 \cdot 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2xy^2}{x^2(2y)}$$

$$= \frac{-y}{x}$$

$$m = \left. \frac{dy}{dx} \right|_{(3,-1)} = \frac{-(-1)}{3} = \frac{1}{3}$$

$$\text{E.O.T.L.} : y + 1 = \frac{1}{3}(x - 3)$$

Problem 10. (10 points) Show that $f(x) = 2x - \cos(x)$ has exactly one zero in the interval $[-\pi, \pi]$.

a. Show that $f(x)$ has a zero.

$$f(-\pi) = -2\pi - (-1) < 0$$

$$f(\pi) = 2\pi - 1 > 0$$

As f is continuous, the IVT gives an x_0 in $[-\pi, \pi]$ such that $f(x_0) = 0$

b. Use Rolle's Theorem to show that it has exactly one zero.

$$f'(x) = 2 + \sin(x), \text{ and, so, } f'(x) > 0.$$

If f had another zero, other than x_0 , then as f is differentiable, Rolle's theorem would give a c such that

$$f'(c) = 0.$$

Since $f'(x) > 0$, this can't happen.

"This" meaning more than one zero.

Problem 11. (18 points) Evaluate the following integrals.

a. $\int_0^1 2x\sqrt{x^2+3} dx$

$u = x^2 + 3, \quad du = 2x dx$

$\hookrightarrow \int_3^4 \sqrt{u} du$

~~$\hookrightarrow \frac{2}{3} du = \frac{2}{3} dx$~~

$= \frac{u^{3/2}}{3/2} \Big|_3^4 = \frac{2}{3} (4^{3/2} - 3^{3/2})$

b. $\int \sin^2(x) \cos(x) dx$

$u = \sin(x), \quad du = \cos(x) dx$

$\hookrightarrow \int u^2 du = \frac{u^3}{3} + C$

$= \frac{\sin^3(x)}{3} + C$

c. Find $f(x)$ such that $f'(x) = \frac{2}{x^2}$ and $f(1) = 0$.

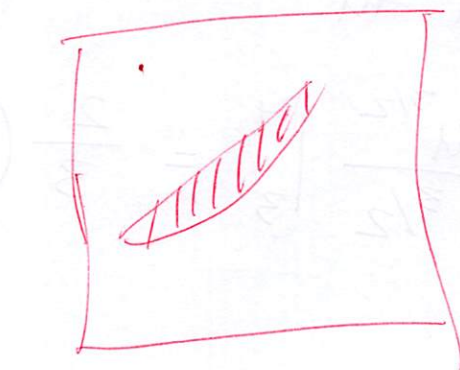
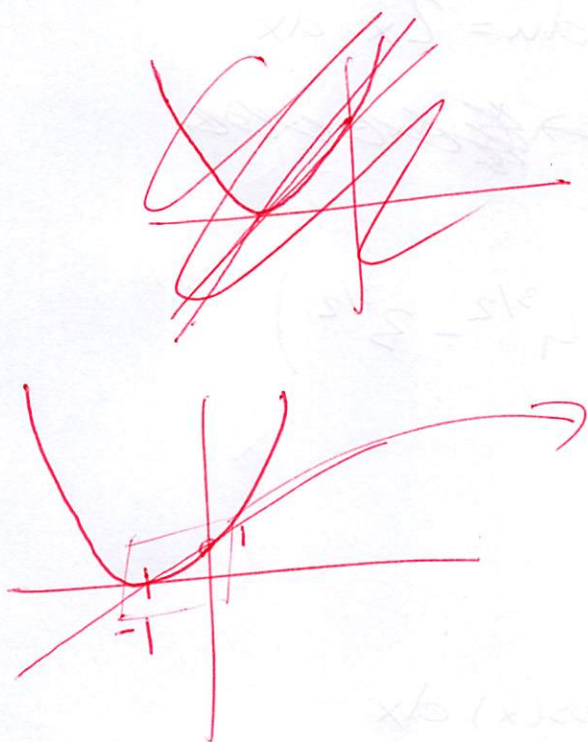
$f'(x) = 2x^{-2}, \quad f(x) = -2x^{-1} + C$

$0 = f(1) = -2 + C$

$\Rightarrow C = 2 \Rightarrow f(x) = \frac{-2}{x} + 2$

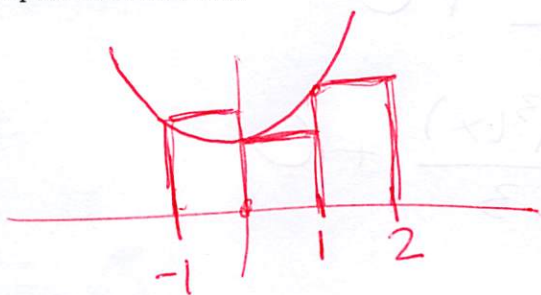
Problem 12. (6 points) Setup an integral for the area between the curve $y = x^2 + 2x + 1$ and the line $y = x + 1$. You do not need to evaluate the integral.

$$= (x+1)^2$$



$$\int_{-1}^0 (x+1) - (x^2 + 2x + 1) dx$$

Problem 13. (6 points) Estimate $\int_{-1}^2 (x^2 + 1) dx$ with a Riemann sum using left endpoints of 3 equal subintervals.



Estimate: $(-1)^2 + 1 = 1$

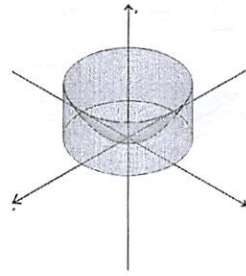
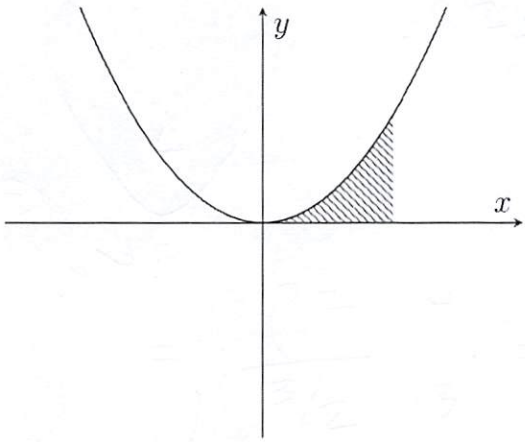
$$+ (0)^2 + 1 = 1$$

$$+ (1)^2 + 1 = 2$$

$$= 2 + 1 + 2$$

$$= 5$$

Problem 14. (10 points) Consider the region between $y = x^2$, the x -axis and the line $x = 1$. Find the volume of the solid that is formed by rotating that region around the y -axis.



$$2\pi \int_0^1 \underbrace{x^2}_{\substack{\text{shell} \\ \text{height}}} \cdot \underbrace{x}_{\substack{\text{shell} \\ \text{radius}}} dx$$

$$\begin{aligned} &\rightarrow 2\pi \left(\frac{x^4}{4} \Big|_0^1 \right) \\ &= 2\pi \left(\frac{1}{4} - \frac{0}{4} \right) \end{aligned}$$

$$\boxed{= \frac{\pi}{2}}$$