Math 241, Spring 2017, Final Exam

Name and section number:

Instructor name:

Solutions
by D. Yuen

Question	Points	Score
1	16	
2	8	
3	16	
4	6	
5	6	
6	7	
7	8	
8	10	
9	12	
10	15	
11	4	
12	6	
13	8	
14	8	
Total:	130	

- You may not use notes or electronic devices on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work.
- You do **not** need to simplify your answers.
- Good luck!

1. Calculate the following limits. **Do not** use L'Hospital's rule. If the limit is positive or negative infinity, say which.

(a) (4 points) $\lim_{x\to\infty} \frac{7-4x-x^4}{2(x^2-2)^2}$.

Type $\frac{\infty}{\infty}$ Divide by highest power of denominator

$$\lim_{x\to\infty} \frac{\frac{7-4x-x^4}{x^4}}{2\frac{(x^2-2)}{x^2}\frac{(x^2-2)}{x^2}} = \lim_{x\to\infty} \frac{\frac{7}{x^4}-\frac{4}{x^3}-1}{2(1-\frac{2}{x^2})(1-\frac{2}{x^2})} = \frac{0-0-1}{2(1-0)(1-0)} = \boxed{-\frac{1}{2}}$$

(b) (4 points) $\lim_{x\to 1^+} \frac{x^2-1}{(x-1)^3}$.

Type $\frac{0}{0}$

$$= \lim_{x\to 1+} \frac{(x+1)(x-1)}{(x-1)^3} = \lim_{x\to 1+} \frac{x+1}{(x-1)^2}$$

Type $\frac{2}{0^+}$

$$= \boxed{\infty}$$

(c) (4 points) $\lim_{x\to 2} \frac{\sqrt{x+7}-3}{x-2} \cdot \frac{\sqrt{x+7}+3}{\sqrt{x+7}+3}$

Type $\frac{0}{0}$

$$= \lim_{x\to 2} \frac{(x+7)-9}{(x-2)(\sqrt{x+7}+3)} = \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{x+7}+3)}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{x+7}+3} = \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

(d) (4 points) $\lim_{x\to 0} \frac{\sin 5x}{x(x+1)}$.

Use $\lim_{x\to 0} \frac{\sin 5x}{5x} = 1$

$$= \lim_{x\to 0} 5 \cdot \frac{\sin 5x}{5x} \cdot \frac{1}{x+1} = 5 \cdot 1 \cdot \frac{1}{0+1} = \boxed{5}$$

2. (a) (6 points) **Using the definition of the derivative as a limit**, compute $f'(0)$ if $f(x) = \frac{1}{2x+1}$. (Warning: you will get no credit if you use the rules of differentiation).

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{1}{2h+1} - \frac{1}{1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2h+1} - \frac{2h+1}{2h+1}}{\frac{h}{1}} = \lim_{h \to 0} \frac{-2h}{2h+1} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-2}{2h+1} = \frac{-2}{0+1} = \boxed{-2}$$

(b) (2 points) The limit $\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$ represents the derivative of some function g at some point a. What is g and what is a?

$$\lim_{h \to 0} \frac{\sqrt{9+h} - \sqrt{9}}{h} = g'(9) \quad \text{where } g(x) = \sqrt{x}$$

$$\uparrow$$
$$a = 9$$

3. Differentiate the following functions. You do not need to simplify your answers.

(a) (4 points) $f(x) = \frac{5}{x^7} - 2x^3 + \sqrt{x} + 7\pi^2$

$$f(x) = 5x^{-7} - 2x^3 + x^{\frac{1}{2}} + 7\pi^2$$

$$f'(x) = -35x^{-8} - 6x^2 + \frac{1}{2}x^{-\frac{1}{2}} + 0$$

(b) (4 points) $g(x) = \frac{x^2(x^3+1)}{2-x^5} = \frac{x^5 + x^2}{2 - x^5}$

$$g'(x) = \frac{(5x^4+2x)(2-x^5) - (x^5+x^2)(-5x^4)}{(2-x^5)^2}$$

(c) (4 points) $h(x) = (1 + \sin(7x^2))^3$

$$h'(x) = 3(1+\sin(7x^2))^2(0 + \cos(7x^2) \cdot 14x)$$

(d) (4 points) $R(x) = \int_0^{3x} (1+t^3)^5 dt$

$$R'(x) = (1 + (3x)^3)^5 \cdot 3$$

4. (6 points) Use linear approximation and the fact that $\frac{1}{100} = 0.01$ to find an approximation to $\frac{1}{102}$.

Set $f(x) = \frac{1}{x}$

Linear approximat at $a = 100$ yields

use $f'(x) = -\frac{1}{x^2}$ $\quad f'(100) = -\frac{1}{10000} = -.0001$

$f(100) = \frac{1}{100} = .01$

$$L(x) = f(100) + f'(100)(x-100)$$
$$= .01 + (-.0001)(x-100).$$

Then $\frac{1}{102} \approx L(102) = .01 + (-.0001)(2)$

$$= .01 - .0002 = .0098$$

5. (6 points) Find an equation for the tangent line to the graph of $x^4 + x^2y + y^3 = 3$ at the point $(1,1)$.

Implicit differentiation. $\frac{d}{dx}$ both sides

$$4x^3 + 2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

$$(x^2+3y^2)\frac{dy}{dx} = -4x^3 - \tfrac{3}{2}xy$$

$$\frac{dy}{dx} = \frac{-4x^3 - 2xy}{x^2+3y^2}$$

At $x=1, y=1$ $\quad \frac{dy}{dx} = \frac{-4-2}{1+3} = \frac{-8}{4} = -2$

Tangent line: $\quad y - 1 = -\frac{3}{2}(x-1).$

6. Consider the equation $1 + x = x^3$.

(a) (5 points) Explain why the equation has a solution in the interval $[1, 2]$. State the theorems you use in your explanation.

Set $f(x) = x^3 - x - 1$.

f is continuous on $[1,2]$.

$f(1) = 1 - 1 - 1 = -1$

$f(2) = 8 - 2 - 1 = 5$.

Since $f(1) < 0 < f(2)$,

by the Intermediate Value Theorem,

$f(c) = 0$ for some $1 < c < 2$.

So $x^3 = x + 1$ has a solution, this c.

(b) (2 points) Explain why the equation can't have two solutions in the interval $[1, 2]$. State the theorems you use in your explanation.

If $f(c) = f(c_2) = 0$ for some $1 \leq c < c_2 \leq 2$,

then because f is differentiable on $[1,2]$,

then $f'(d) = 0$ for some $c < d < c_2$

by Rolle's Theorem.

But $f'(x) = 3x^2 - 1 \geq 3(1)^2 - 1 = 2$

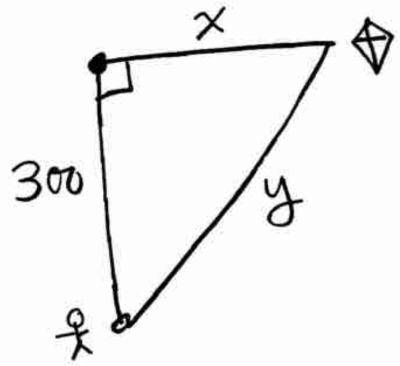
for $x \geq 1$.

Thus $f'(x) \neq 0$ for all $1 \leq x \leq 2$.

So $f'(d) = 0$ is not possible for $c < d < c_2$.

So there cannot be two solutions in $[1,2]$.

7. (8 points) A person flies a kite at a height of 300 feet. The wind carrying the kite moves it away from the person horizontally at a speed of 25 feet per second. What is the rate of change of the length of the kite string (that is - the distance from the person to the kite), when the kite is 500 feet away from the person?

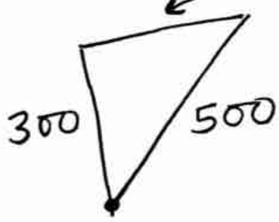


Given $\frac{dx}{dt} = 25$ ft/s $\boxed{\text{units ft, s}}$

$\frac{dy}{dt} = ?$

Evaluate when

solve = 400



plug in
$x = 400$
$y = 500$

$$x^2 + 300^2 = y^2$$

$\frac{d}{dt}$ both sides

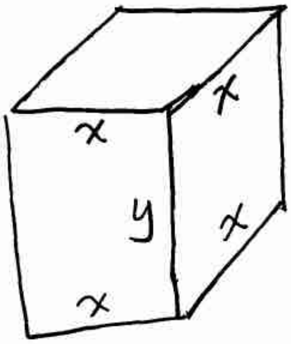
$$2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$$

$$2(400)(25) = 2(500)\frac{dy}{dt}$$

$$\boxed{\frac{2(400)(25)}{2(500)} \text{ ft/s}} = \frac{dy}{dt}$$

$$20 \text{ ft/s} = \frac{dy}{dt}$$

8. (10 points) A rectangular box has a base that is a square. The perimeter of the base plus the height of the box is equal to 3 feet. What is the largest possible volume for such a box, and what are its dimensions? Justify your answer.



Constraint is
$4x + y = 3 \Rightarrow y = 3 - 4x$

Maximize volume
$V = x^2 y$

$V = x^2(3 - 4x)$ Domain $0 \le x \le \frac{3}{4}$ because $3 - 4x \ge 0$

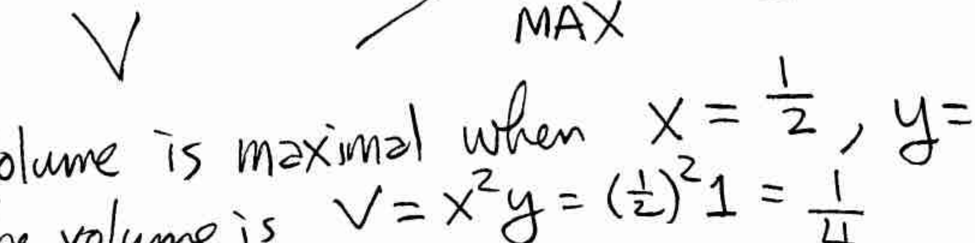
$V = 3x^2 - 4x^3$

$\frac{dV}{dx} = 6x - 12x^2 \overset{\text{set}}{=} 0$

$6x(1 - 2x) = 0$

$x = 0 \qquad x = \frac{1}{2}$

1ST derivative diagram



Volume is maximal when $x = \frac{1}{2}$, $y = 3 - 4(\frac{1}{2}) = 1$,
The volume is $V = x^2 y = (\frac{1}{2})^2 1 = \frac{1}{4}$.

9. Let $f(x) = 3x^5 - 5x^3$.

(a) (2 points) find the critical points of f.

$$f'(x) = 15x^4 - 15x^2 \overset{\text{set}}{=} 0$$

$$15x^2(x^2 - 1) = 0$$

$x = 0,\ 1,\ -1$ are critical points

(b) (2 points) Classify the critical points of f as local maxima, local minima, or neither.

$f'(x)$: + + + 0 − − 0 − − 0 + +

x: −1, 0, 1

f: ↗ LOCAL MAX → NEITHER → LOCAL MIN ↗

(c) (2 points) Find the intervals where f is increasing.

Increasing on $(-\infty, 1]$ and on $[1, \infty)$.

$$f(x) = 3x^5 - 5x^3$$

(d) (2 points) Find the maximal and minimal values of f in $[-2, 0]$.

Compare critical points and endpoints

$x = -1 \quad f(-1) = -3 + 5 = 2 \leftarrow$ MAX

$x = -2 \quad f(-2) = -96 + 45 = -51 \leftarrow$ MIN

$x = 0 \quad f(0) = 0$

(e) (2 points) Find the intervals where f in concave up.

$$f''(x) = 60x^3 - 30x = 0$$

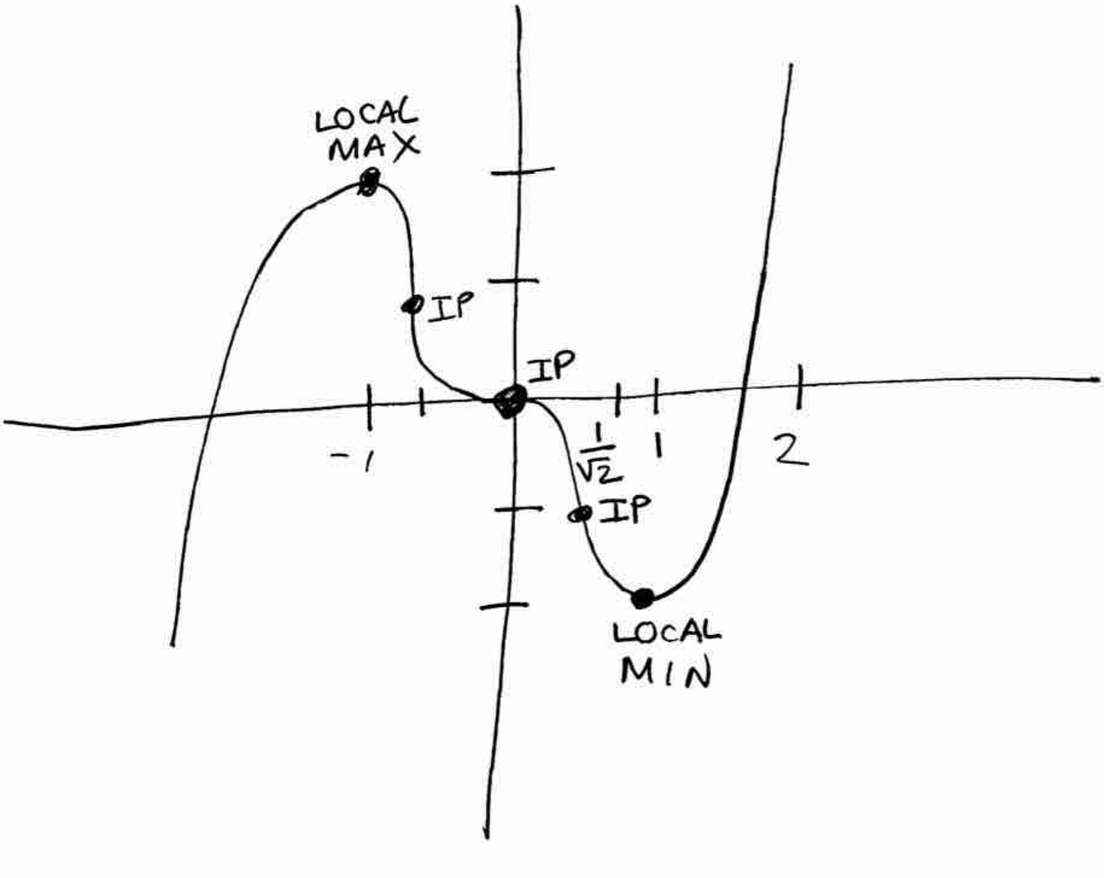
$$30x(2x^2 - 1) = 0$$

$$x = 0 \qquad x = \pm \frac{1}{\sqrt{2}}$$

$f''(x)$	$- - -$	0	$+ +$	0	$- -$	0	$+ +$
x		$-1/\sqrt{2}$		0		$1/\sqrt{2}$	

concave up on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $(0, \infty)$

(f) (2 points) Give a rough sketch of the graph of $y = f(x)$.



10. Compute each of the following.

(a) (5 points) $\int_0^{\frac{\pi}{2}} \sin(x)\cos^5(x)dx$

Sub $u = \cos x$
$du = -\sin x \, dx$
$x = \frac{\pi}{2} \Rightarrow u = \cos\frac{\pi}{2} = 0$
$x = 0 \Rightarrow u = \cos 0 = 1$

$= \int_1^0 u^5(-du)$

$= -\frac{1}{6}u^6 \Big|_1^0$

$= -0 + \frac{1}{6} = \boxed{\frac{1}{6}}$

(b) (5 points) $\int \frac{x^2-1}{\sqrt{(x^3-3x)}}dx$

$u = x^3 - 3x$
$du = (3x^2-3)dx$
$du = 3(x^2-1)dx$

$= \frac{1}{3}\int \frac{3(x^2-1)dx}{\sqrt{x^3-3x}}$

$= \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\cdot 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}(x^3-3x)^{1/2} + C$

(c) (5 points) Find the function $F(x)$ given that $F'(x) = x^2 + 4x + 5$ and $F(1) = 2$.

$F(x) = \frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C$
$F(x) = \frac{1}{3}x^3 + 2x^2 + 5x + C$

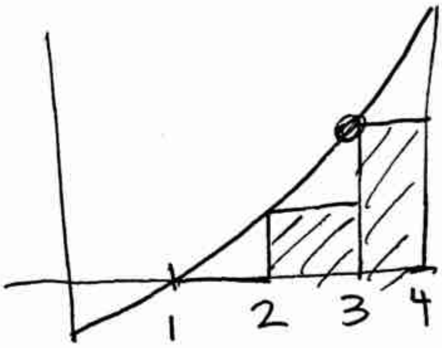
At $x=1$:

$2 = F(1) = \frac{1}{3} + 2 + 5 + C$
$-\frac{1}{3} - 5 = C$
$-\frac{16}{3} = C$

$F(x) = \frac{1}{3}x^3 + 2x^2 + 5x - \frac{16}{3}$

11. Let $f(x) = x^2 - 1$. Partition the interval $[1, 4]$ into 3 equal parts.

(a) (2 points) Calculate a Riemann sum for f using the left endpoint of each interval.



$\Delta x = \frac{4-1}{3} = 1$

$R_3 = (f(1) + f(2) + f(3))\Delta x = (0 + 3 + 8)1 = \boxed{11}$

(b) (2 points) Is the Riemann sum you calculated in the previous part more or less than $\int_1^4 (x^2 - 1)dx$? Explain your answer.

Less, as the diagram shows.

$\int_1^4 (x^2-1)dx$ = area under x^2-1 between 1 and 4

The Riemann sum in this case is also the lower sum.

12. For each of the following, answer True or False. No further explanation is required.

(a) (2 points) Every differentiable function is also continuous.

True

(b) (2 points) The function $F(x) = \int_0^x \frac{1}{1+t^2+t^4} dt$ is increasing.

True

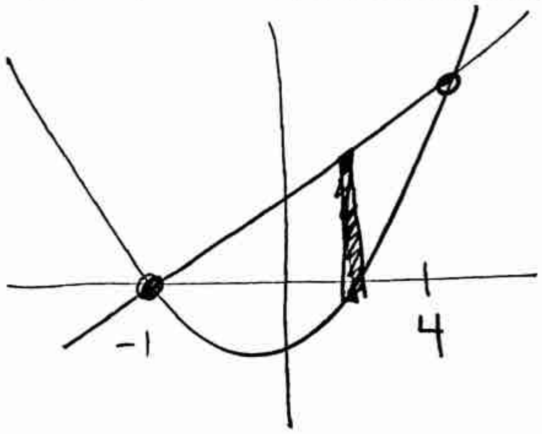
because $F'(x) = \frac{1}{1+x^2+x^4}$ is always > 0.

(c) (2 points) If $f'(1) = 0$ and $f''(1) = 0$ then f cannot achieve a local maximum at 1.

False

$f(x) = -(x-1)^4$ is such an example.

13. (8 points) Calculate the area bounded by the graphs of $y = x^2 - 1$ and $y = 3x + 3$.



Solve for intersection points

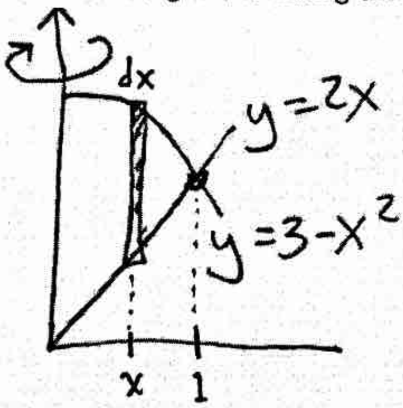
$$x^2 - 1 = 3x + 3$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1)$$
$$x = 4, -1$$

$$A = \int_{-1}^{4} (3x + 3 - (x^2 - 1))\,dx$$
$$= \int_{-1}^{4} (3x + 4 - x^2)\,dx$$
$$= \frac{3}{2}x^2 + 4x - \frac{1}{3}x^3 \Big|_{-1}^{4}$$
$$= \frac{3}{2}(16) + 16 - \frac{64}{3} - \left(\frac{3}{2} - 4 + \frac{1}{3}\right)$$

intersection: $2x = 3 - x^2$
$x^2 + 2x - 3 = 0$
$(x+3)(x-1) = 0$
$x = 1, -3$

14. Consider the region R bounded by the graphs of $y = 2x$, $y = 3 - x^2$ and $x \geq 0$.

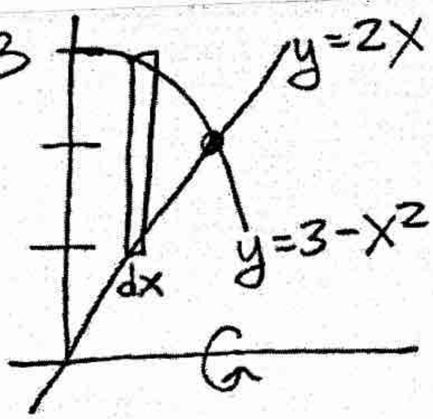
(a) (4 points) The region R is rotated about the y-axis. Set up, but do not evaluate an integral describing the volume of the resulting shape. You may use any method you like.



Get shells

$$V = 2\pi \int_0^1 \underbrace{(x-0)}_{\text{radius}} \underbrace{(3 - x^2 - 2x)}_{\text{height}}\, dx$$

(b) (4 points) The region R is rotated about the x-axis. Set up, but do not evaluate an integral describing the volume of the resulting shape. You may use any method you like.



Use vertical rectangles, dx-problem
Get washers.

$$V = \int_0^1 \Big(\underbrace{(3-x^2)^2}_{\text{outer radius}} - \underbrace{(2x)^2}_{\text{inner radius}} \Big)\, dx$$