Math 241 Final Exam
Spring 2018
5/9/2018
Time limit: 120 minutes

Name: ____________________

Please read carefully

- No calculators or notes are allowed.
- Show your work.
- When applicable, indicate your final answer by drawing a box around it.
- Circle your instructor and section number:

Robertson 1	Lyons 4	Hadari 7
Antin 2	Lyons 5	Harron 8
Antin 3	Hadari 6	Harron 9

Grade Table (for instructor use only)

Page	Points	Score
2	24	
3	24	
4	32	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	6	
12	16	
13	16	
14	10	
15	12	
16	12	
Total:	200	

1. Calculate the derivatives of the following functions. You do not have to simplify your answer.

(a) (8 points) $y = \frac{2}{x^3} + \frac{x^3}{2}$

$y = 2x^{-3} + \frac{1}{2}x^3$

$\frac{dy}{dx} = \boxed{-6x^{-4} + \frac{3}{2}x^2}$

(b) (8 points) $y = \frac{1}{2+x^2}$

Quotient Rule

$\frac{dy}{dx} = \frac{(2+x^2)\,{}^{d}/_{dx}(1) - (1)\,{}^{d}/_{dx}(2+x^2)}{(2+x^2)^2} = \boxed{\frac{(2+x^2)(0) - (1)(2x)}{(2+x^2)^2}}$

Product and Chain

$y = (2+x^2)^{-1} \Rightarrow \frac{dy}{dx} = -(2+x^2)^{-2}\frac{d}{dx}(2+x^2) = \boxed{-(2+x^2)^{-2}(2x)}$

(c) (8 points) $y = \cos\left(\frac{x}{2}\right)\tan x$

$\frac{dy}{dx} = \left(\frac{d}{dx}\cos\left(\frac{x}{2}\right)\right)\tan x + \cos\left(\frac{x}{2}\right)\left(\frac{d}{dx}(\tan x)\right)$

$= -\sin\left(\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right)\tan x + \cos\left(\frac{x}{2}\right)\sec^2 x$

$= \boxed{-\sin\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)\tan x + \cos\left(\frac{x}{2}\right)\sec^2 x}$

(Continued on next page...)

(d) (8 points) $y = \sin(3\sqrt{1-x})$

$$y = \sin(3(1-x)^{1/2})$$

$$\frac{dy}{dx} = \cos(3(1-x)^{1/2})\frac{d}{dx}(3(1-x)^{1/2}) = \cos(3(1-x)^{1/2})\, 3\left(\frac{1}{2}(1-x)^{-1/2}\frac{d}{dx}(1-x)\right)$$

$$= \boxed{\cos(3(1-x)^{1/2})\left(\frac{3}{2}(1-x)^{-1/2}(-1)\right)}$$

(e) (8 points) $y = \dfrac{\sqrt[3]{x}-1}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)\, d/dx\, (x^{1/3}-1) - (\sqrt[3]{x}-1)\, d/dx\, (x-1)}{(x-1)^2}$$

$$= \boxed{\frac{(x-1)(1/3\, x^{-2/3}) - (\sqrt[3]{x}-1)(1)}{(x-1)^2}}$$

(f) (8 points) $y = \displaystyle\int_x^0 \sqrt{1+t^2}\, dt$

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^0 \sqrt{1+t^2}\, dt = -\frac{d}{dx}\int_0^x \sqrt{1+t^2}\, dt = \boxed{-\sqrt{1+x^2}} \quad \text{(by FTC)}$$

2. For each of the following limits, if it exists, compute it. If the limit is infinite, say whether it is ∞ or $-\infty$. Do not use L'Hospital's rule, even if you know it.

(a) (8 points) $\lim_{x\to 2} \frac{x^3-6x+4}{x+2}$

$$\lim_{x\to 2} \frac{x^3-6x+4}{x+2} = \frac{(2)^3-6(2)+4}{2+2} = \frac{8-12+4}{4} = \frac{0}{4} = \boxed{0}$$

(b) (8 points) $\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$

type: "$\frac{0}{0}$"

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \cdot \frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} = \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x}-3)(\sqrt{x}+\sqrt{3})} = \lim_{x\to 3} \frac{\cancel{(x-3)}(\sqrt{x}+3)}{\cancel{x-3}}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3}) = \sqrt{3}+\sqrt{3} = \boxed{2\sqrt{3}}$$

(c) (8 points) $\lim_{x\to\infty} \frac{x^3+1}{3-x^3}$

type: "$\frac{\infty}{-\infty}$"

$$\lim_{x\to\infty} \frac{x^3+1}{3-x^3} = \lim_{x\to\infty} \frac{x^3/x^3+1/x^3}{3/x^3-x^3/x^3} = \lim_{x\to\infty} \frac{1+1/x^3}{3/x^3-1} = \frac{1+0}{0-1} = \boxed{-1}$$

(d) (8 points) $\lim_{x\to 0} \frac{1}{x^{2/3}}$

$$\lim_{x\to 0^-} \frac{1}{x^{2/3}} \to \frac{1}{0^+} \to \infty$$

$$\lim_{x\to 0^+} \frac{1}{x^{2/3}} \to \frac{1}{0^+} \to \infty$$

so $\lim_{x\to 0} \frac{1}{x^{2/3}} \to \boxed{\infty}$

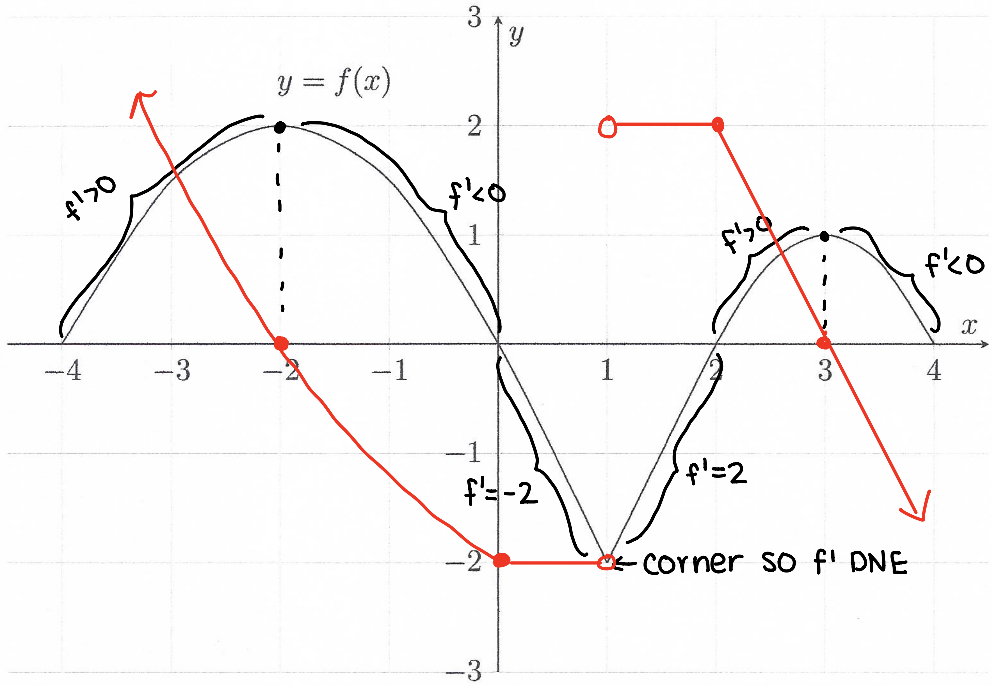
3. (8 points) Using the **limit definition of the derivative**, find $f'(x)$ if $f(x) = \dfrac{1}{x+1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h} \cdot \frac{(x+h+1)(x+1)}{(x+h+1)(x+1)}$$

$$= \lim_{h\to 0} \frac{x+1-(x+h+1)}{h(x+h+1)(x+1)} = \lim_{h\to 0} \frac{\cancel{x}+\cancel{1}-\cancel{x}-h-\cancel{1}}{h(x+h+1)(x+1)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+h+1)(x+1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+0+1)(x+1)} = \boxed{\frac{-1}{(x+1)^2}}$$

4. (8 points) The following is the graph of a function f on the domain $[-4, 4]$.

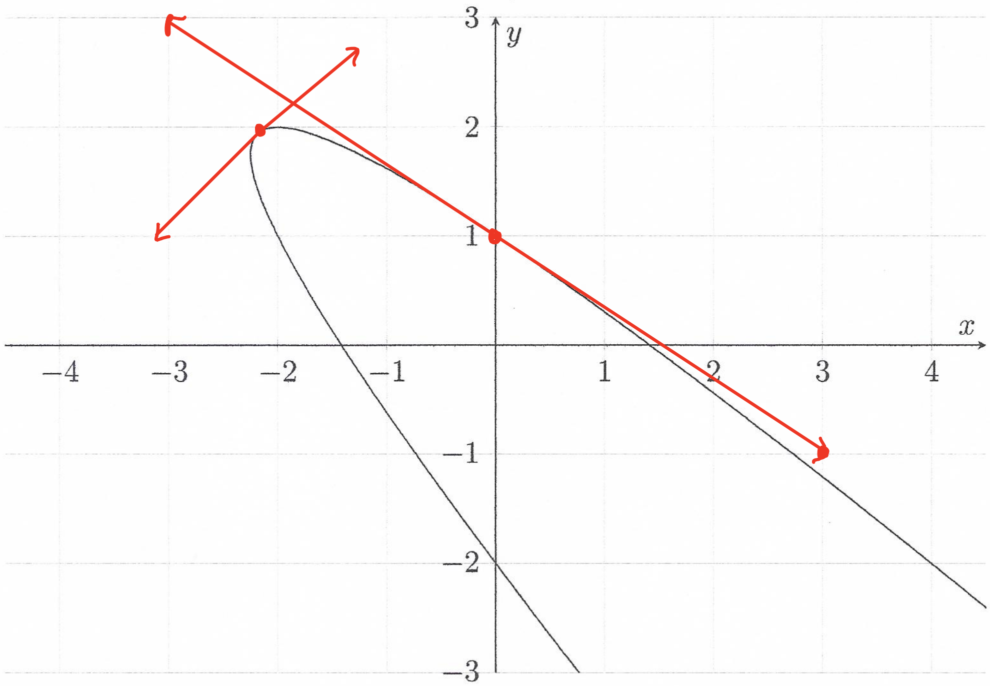


On the same set of axes, sketch the graph of the derivative f'. If there are discontinuities, indicate them clearly.

5. The equation

$$x^2 + y^2 + 2xy + y = 2$$

describes the following curve in the plane:



(a) (4 points) Find $\frac{dy}{dx}$ at the point $(0, 1)$.

$2x+2y\frac{dy}{dx}+2y+2x\frac{dy}{dx}+\frac{dy}{dx}=0$

plug in $x=0, y=1$

$2\frac{dy}{dx}+2+\frac{dy}{dx}=0 \Rightarrow 3\frac{dy}{dx}=-2 \Rightarrow \frac{dy}{dx}=\boxed{\frac{-2}{3}}$

(b) (2 points) Find an equation for the tangent line to this curve at the point $(0, 1)$.

$m=\frac{-2}{3}$

$y-1=\frac{-2}{3}(x-0) \Rightarrow \boxed{y=\frac{-2}{3}x+1}$

(c) (2 points) On the graph above, sketch the tangent lines to the curve at the points $(0, 1)$ and $(-2, 2)$.

6. You're pumping air into a spherical balloon at a steady rate of $5\,\text{cm}^3/\text{s}$.

(The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ while its surface area is $4\pi r^2$.)

(a) (4 points) At what rate is its radius increasing (in cm/s) when its radius is 3 cm?

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Given: $r = 3$ and $\frac{dV}{dt} = 5$

$$5 = 4\pi(3)^2 \frac{dr}{dt} = 36\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \boxed{\frac{5}{36\pi}\ \text{cm/s}}$$

(b) (4 points) At what rate is its surface area increasing (in cm^2/s) when its radius is 3 cm?

$$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

Given: $\frac{dr}{dt} = \frac{5}{36\pi}$ (from (a)) and $r = 3$

$$\frac{dS}{dt} = 8\pi(3)\left(\frac{5}{36\pi}\right) = 8\pi\left(\frac{5}{12\pi}\right) = \frac{40}{12} = \boxed{\frac{10}{3}\ \text{cm}^2/\text{s}}$$

7. (8 points) A designer wants to make a new line of bookcases. They want to make at least 10 of them and not more than 40. They predict that the average cost of producing x bookcases is

$$A(x) = 42\left(x + \frac{400}{x}\right) \text{ dollars.}$$

Find the number of bookcases that minimizes the average cost.

$A(x) = 42(x + 400x^{-1})$

$A'(x) = 42(1 - 400x^{-2})$

$42(1-400x^{-2}) = 0 \Rightarrow 1 - 400x^{-2} = 0$

$\Rightarrow 1 - \frac{400}{x^2} = 0 \Rightarrow 1 = \frac{400}{x^2} \Rightarrow x^2 = 400 \Rightarrow x = 20$

check this is a min

A': on $[10, 20]$: $-$; on $[20, 40]$: $+$ (or check endpts)

$\boxed{x = 20 \text{ book cases}}$

8. Let

$$f(x) = \frac{1}{1+x^2}.$$

Then

$$f'(x) = -\frac{2x}{(1+x^2)^2} \quad \text{and} \quad f''(x) = \frac{6x^2-2}{(1+x^2)^3}.$$

(a) (4 points) Find the intervals on which the graph of f is increasing and those on which it is decreasing. Find the local minimums and maximums, if there are any, and determine which of them are absolute.

$f'(x)$ never undefined

$f'(x)=0 \Rightarrow \frac{-2x}{(1+x^2)^2}=0 \Rightarrow -2x=0 \Rightarrow x=0$

f': $+$ | $-$ at 0

increasing on $(-\infty,0)$ decreasing on $(0,\infty)$

relative max at $x=0$ (and absolute) point $(0,f(0))=(0,1)$

(b) (4 points) Find the intervals on which the graph of f is concave up and those on which it is concave down. Find the points of inflection, if any exist.

$f''(x)$ never undefined

$f''(x)=0 \Rightarrow \frac{6x^2-2}{(1+x^2)^3}=0 \Rightarrow 6x^2-2=0 \Rightarrow 6x^2=2 \Rightarrow x^2=\frac{1}{3} \Rightarrow x=\pm\frac{1}{\sqrt{3}}$

f'': $+$ | $-$ | $+$ at $\frac{-1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$

concave up on $(-\infty,-1/\sqrt{3}),(1/\sqrt{3},\infty)$
concave down on $(-1/\sqrt{3},1/\sqrt{3})$

inflection points at $x=\pm 1/\sqrt{3}$

$f\left(\frac{-1}{\sqrt{3}}\right)=\frac{1}{1+(-1/\sqrt{3})^2}=\frac{1}{1+1/3}=\frac{1}{4/3}=\frac{3}{4}$ $\quad f\left(\frac{1}{\sqrt{3}}\right)=\frac{1}{1+(1/\sqrt{3})^2}=\frac{1}{1+1/3}=\frac{1}{4/3}=\frac{3}{4}$

inflection points: $\left(\pm\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$

(Continued on next page...)

(c) (2 points) Find the asymptotes, if there are any.

$f(x) = \frac{1}{1+x^2}$ is never undefined $\Rightarrow$ no vertical asymptotes

Horizontal Asymptotes:

$\lim_{x\to\infty} \frac{1}{1+x^2} = 0$, $\lim_{x\to-\infty} \frac{1}{1+x^2} = 0$ so $y=0$

(d) (4 points) Sketch the graph on the axes below. Label the asymptotes, maximums, minimums, and points of inflection, if there are any.

9. Compute the following integrals.

(a) (8 points) $\int_0^{\pi/2} \cos(2\theta)\, d\theta$

$u = 2\theta \Rightarrow du = 2d\theta \Rightarrow d\theta = \frac{du}{2}$

bounds: $\theta = 0 \Rightarrow u = 0$, $\theta = \pi/2 \Rightarrow u = \pi$

$\int_0^{\pi/2} \cos(2\theta) d\theta = \int_0^{\pi} \cos(u) \frac{du}{2} = \frac{1}{2}\int_0^{\pi} \cos u \, du = \frac{1}{2}\sin u \Big|_0^{\pi}$

$= \frac{1}{2}\sin(\pi) - \frac{1}{2}\sin(0) = 0 - 0 = \boxed{0}$

(b) (8 points) $\int (\sqrt[3]{2} - x^{2/3})\, dx$

$\int (\sqrt[3]{2} - x^{2/3}) dx = \sqrt[3]{2}x - \frac{x^{5/3}}{5/3} + C = \boxed{\sqrt[3]{2}x - \frac{3}{5}x^{5/3} + C}$

(c) (8 points) $\int \frac{\sin 3x}{\cos^7 3x}\,dx$

$u = \cos(3x) \Rightarrow du = -\sin(3x)(3)\,dx \Rightarrow dx = \frac{-du}{3\sin(3x)}$

$\int \frac{\sin(3x)}{\cos^7(3x)}\,dx = \int \frac{\cancel{\sin(3x)}}{u^7}\cdot\frac{-du}{3\cancel{\sin(3x)}} = -\frac{1}{3}\int u^{-7}du = -\frac{1}{3}\frac{u^{-6}}{-6} + C = \frac{u^{-6}}{18} + C$

$= \boxed{\frac{\cos^{-6}(3x)}{18} + C}$

(d) (8 points) $\int_0^{\sqrt{8}} \frac{x}{\sqrt{1+x^2}}\,dx$

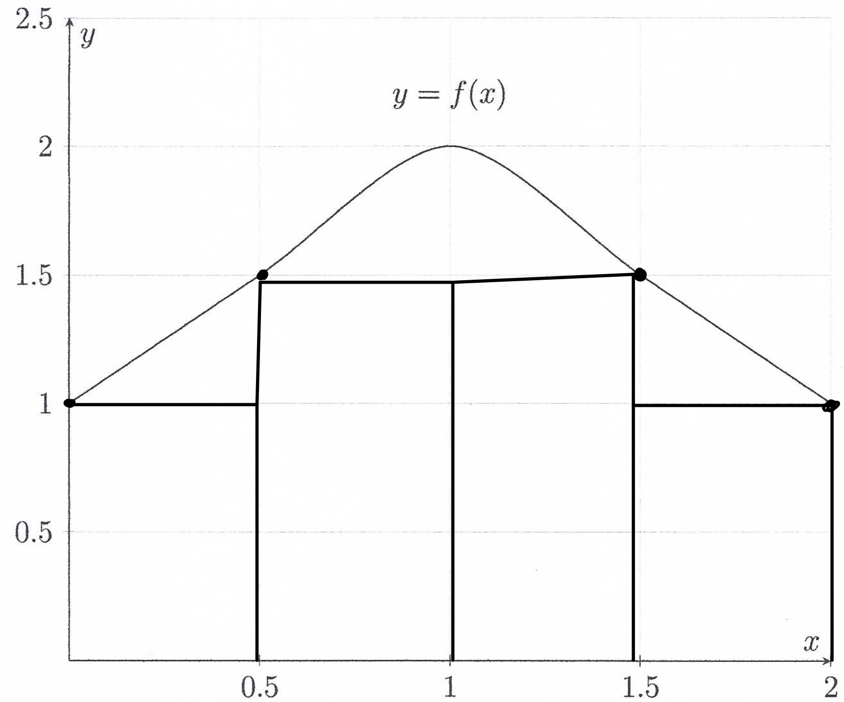
$u = 1 + x^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

new bounds: $x = 0 \Rightarrow u = 1$

$x = \sqrt{8} \Rightarrow u = 9$

$\int_0^{\sqrt{8}} \frac{x}{\sqrt{1+x^2}}\,dx = \int_1^9 \frac{\cancel{x}}{\sqrt{u}}\,\frac{du}{2\cancel{x}} = \frac{1}{2}\int_1^9 u^{-1/2}du = \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right)\Big|_1^9 = \sqrt{u}\,\Big|_1^9 = \sqrt{9} - \sqrt{1} = 3 - 1 = \boxed{2}$

10. The following is the graph of a function f on the interval $[0, 2]$:



You are asked to compute an approximation to $\int_0^2 f(x)\,dx$ by means of a **lower Riemann sum using 4 subintervals of equal width**.

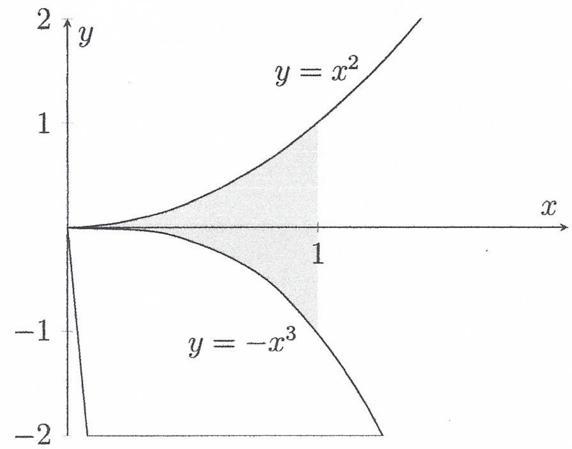
(a) (5 points) On the graph above, draw the rectangles you should use.

(b) (5 points) Compute the Riemann sum.

$\frac{1}{2}f(0)+\frac{1}{2}f(0.5)+\frac{1}{2}f(1.5)+\frac{1}{2}f(2)$

$=\frac{1}{2}(1)+\frac{1}{2}(1.5)+\frac{1}{2}(1.5)+\frac{1}{2}(1) = 1+1.5 = \boxed{2.5}$

11. Consider the region enclosed by the graphs of $y = x^2$ and $y = -x^3$ between $x = 0$ and $x = 1$:

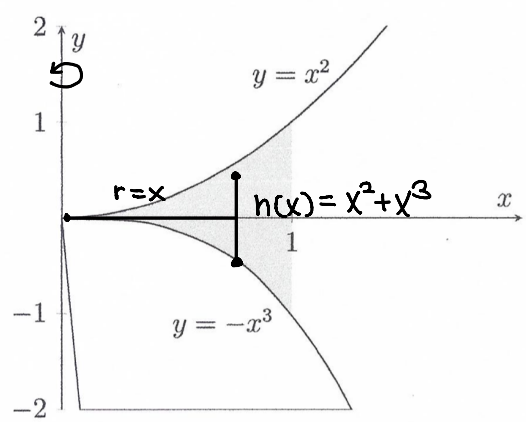


(a) (6 points) Compute the area of this region.

$$A = \int_0^1 (x^2 - (-x^3))\,dx = \int_0^1 (x^2 + x^3)\,dx = \frac{x^3}{3} + \frac{x^4}{4}\Big|_0^1$$
$$= \left(\frac{1}{3} + \frac{1}{4}\right) - (0+0) = \frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \boxed{\frac{7}{12}}$$

(b) (6 points) Set up, but **do not evaluate**, an integral that gives the volume of the solid obtained by revolving this region around the y-axis.

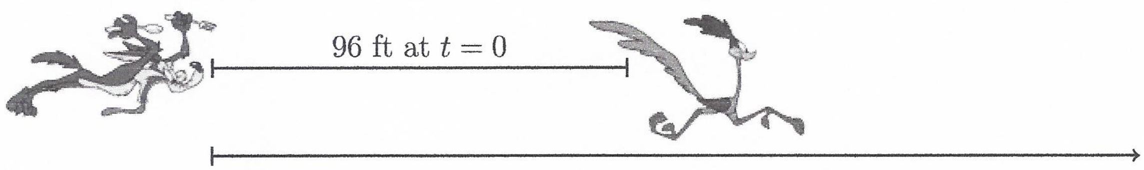
shell method



$$V = \int_0^1 2\pi x (x^2 + x^3)\,dx$$

12. The Road Runner is running in a straight line and the Coyote is chasing after him. Suppose that at time $t = 0$, the Road Runner has a head start of 96 ft. The Road Runner's speed is a constant 4 ft/sec, while the Coyote's is $4t$ ft/sec.

The Coyote The Road Runner



(a) (5 points) Find the Road Runner's position as a function of time.

$S_R(t)$ = position of Road Runner

$S_R(0) = 96$

$V_R(t)$ = velocity of Road Runner = 4

$S_R(t) = \int V_R(t)\,dt = \int 4\,dt = 4t + C$

$S_R(0) = C = 96$

$\Rightarrow \boxed{S_R(t) = 4t + 96}$

(b) (5 points) Find the Coyote's position as a function of time.

$S_C(t)$ = position of Coyote

$S_C(0) = 0$

$V_C(t)$ = velocity of Coyote $= 4t$

$S_C(t) = \int V_C(t)\,dt = \int 4t\,dt = \frac{4t^2}{2} + C = 2t^2 + C$

$S_C(0) = C = 0$

$\Rightarrow \boxed{S_C(t) = 2t^2}$

(c) (2 points) How long would it take for the Coyote to catch up with the Road Runner?

$4t + 96 = 2t^2 \Rightarrow 2t^2 - 4t + 96 = 0$

$\Rightarrow t^2 - 2t - 48 = 0$

$\Rightarrow (t-8)(t+6) = 0$

$\Rightarrow t = 8, -6$

$\Rightarrow t = 8$

$\boxed{8 \text{ seconds}}$