Math 241, Spring 2012
Final Exam

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Instructor:

Instructions: Write legibly. To earn full credit, you must show enough of your work to justify your answers. Turn off and store all of your electronic devices; this includes calculators, cell phones, tablets and music players. All work should be your own.

Problem	Worth	Score
1	20	
2	10	
3	10	
4	20	
5	10	
6	10	
7	20	
8	10	
9	20	
10	10	
11	10	
12	10	
Total	160	

Extra Credit (5 points). Evaluate the following derivative: $\frac{d}{dx}\int_{x^2}^{\sin x} \sqrt{1+t^2}\, dt.$

$= \sqrt{1+\sin^2(x)} \cdot \cos(x) - \sqrt{1+x^4} \cdot 2x$

Problem 1 (20 points). Evaluate each of the following limits or show that they do not exist. **Show your work!**

(a) $\lim_{x \to -2} \frac{x^2 - 4x + 5}{x^2 - 2}$ $= \frac{(-2)^2 - 4(-2) + 5}{(-2)^2 - 2} = \frac{17}{2}$

(b) $\lim_{x \to 2^-} \frac{x^2 - 4}{|x + 2|}$ $= \frac{2^2 - 4}{|2+2|} = \frac{0}{4} = 0$

(c) $\lim_{x \to \infty} \frac{\sin(x^2)}{\sqrt{x}}$

$-1 \leq \sin(x^2) \leq 1$

$\Leftrightarrow \quad \frac{-1}{\sqrt{x}} \leq \frac{\sin(x^2)}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}.$

Since $\lim_{x \to \infty} \frac{-1}{\sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$,

$\lim_{x \to \infty} \frac{\sin(x^2)}{\sqrt{x}} = 0$ by sandwich Theorem.

(d) $\lim_{x \to 0} \frac{1 - \cos x}{x}$ $\frac{(1 + \cos(x))}{(1 + \cos(x))}$

$= \lim_{x \to 0} \frac{1 - \cos^2(x)}{x(1 + \cos(x))} = \lim_{x \to 0} \frac{\sin^2(x)}{x(1 + \cos(x))}$

$= \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \sin(x) \cdot \frac{1}{1 + \cos(x)}$

$= 1 \cdot 0 \cdot 1$

$= 0$

Problem 2 (10 points). A particle moves in a straight line along the s-axis. At time t, its acceleration is $a(t) = -6t + 2$. Its position and velocity at time $t = 4$ are $s(4) = 1$ and $v(4) = 1$, respectively. Find the position function $s(t)$.

$$V(t) = \int a(t)\,dt = -3t^2 + 2t + C$$

$$1 = V(4) = -3(4)^2 + 2(4) + C$$

$$1 = -48 + 8 + C$$

$$1 = -40 + C \Rightarrow C = 41.$$

$$S(t) = \int v(t)\,dt = \int -3t^2 + 2t + 41$$

$$= -t^3 + t^2 + 41t + C$$

$$1 = S(4) = -(4)^3 + 4^2 + 41(4) + C$$

$$\Rightarrow 1 = -64 + 16 + 164 + C \Rightarrow C = -115$$

So
$$\boxed{S(t) = -t^3 + t^2 + 41t - 115}$$

Problem 3 (10 points). Find an equation for the tangent line to the curve given by

$$2x^2y - 3xy^2 = 16$$

at the point $(-1, 2)$.

$$\frac{d}{dx}(2x^2y - 3xy^2) = \frac{d}{dx}(16)$$

$$\Rightarrow 4xy + 2x^2y' - 3y^2 - 3x \cdot 2y \cdot y' = 0$$

$$\Rightarrow y'(2x^2 - 6xy) = 3y^2 - 4xy$$

$$\Rightarrow y' = \frac{3y^2 - 4xy}{2x^2 - 6xy}$$

Now, $$y'\big|_{(-1,2)} = \frac{3 \cdot 2^2 - 4(-1) \cdot 2}{2 \cdot (-1)^2 - 6(-1) \cdot 2}$$

$$= \frac{20}{14} = \frac{10}{7}$$

and whence, the equation of the tangent line is

$$y - 2 = \frac{10}{7}(x - (-1))$$

Problem 4 (20 points). Find the derivative of each of the following functions.
Do not simplify!

(a) $f(x) = \sqrt{5 + \frac{2}{x^6}}$, $f'(x) = \dfrac{1}{2\sqrt{5+\frac{2}{x^6}}} \cdot \dfrac{-12}{x^7}$

(b) $g(x) = x^7 \tan x$, $g'(x) = 7x^6 \tan(x) + \sec^2(x)\, x^7$

(c) $h(x) = \dfrac{\cos(7 \sin x)}{8 + \sec(2x)}$, ~~$h'(x) = -\sin(7\sin(x)) \cdot 7\cos(x)(8+\sec(2x)) - \sec(2x)\tan(2x)\cdot 2$~~

$$h'(x) = \frac{-\sin(7\sin(x)) \cdot 7\cos(x)\left(8+\sec(2x)\right) - \sec(2x)\tan(2x)\cdot 2\left(\cos(7\sin(x))\right)}{\left(8+\sec(2x)\right)^2}$$

(d) $j(x) = \displaystyle\int_2^x \frac{t^5}{7+t^8}\, dt.$

$$j'(x) = \frac{x^5}{7+x^8}$$

Problem 5 (10 points). Consider the equation $x^3 - 7x + 1 = 0$.

(a) Does this equation have a solution in the interval $[2, 3]$? Justify your answer.

~~As~~ let $f(x) = x^3 - 7x + 1$. Note that $f(x)$ is continuous and differentiable.

Since $f(2) = 2^3 - 7(2) + 1 = -5$ and $f(3) = 3^3 - 7(3) + 1 = 7$ the IVT gives us an x_0 in $[2,3]$ such that $f(x_0) = 0$. This x_0 is a solution to our equation!

(b) Does this equation have more than one solution in this interval?

$f'(x) = 3x^2 - 7$ is > 0 on $[2,3]$.

If there were to be another x_1 in $[2,3]$ with $f(x_1) = 0$, Rolle's theorem would give a c in $(2,3)$ such that $f'(c) = 0$. So, there can't be another solution in $[2,3]$.

Problem 6 (10 points). Using the **limit definition** of the derivative, find the derivative of

$$f(x) = x^2 + 3x + 1$$

at $x = 2$. **To receive credit, you must show your work. It is not acceptable to use differentiation rules.**

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 + 3(2+h) + 1 - 11}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 + 6 + 3h + 1 - 11}{h}$$

$$= \lim_{h \to 0} \frac{h(4 + h + 3)}{h}$$

$$= \lim_{h \to 0} 7 + h = 7$$

Problem 7 (20 points). Below you are given a function $f(x)$ and its first and second derivatives. Use this information to solve the following problems.

$$f(x) = \frac{x^2 - 4}{x^2 + 1} \qquad f'(x) = \frac{10x}{(x^2+1)^2} \qquad f''(x) = \frac{10(1 - 3x^2)}{(x^2+1)^3}$$

(a) Find the global maximum and minimum value of $f(x)$ on the interval $[-2, 3]$. **Show your work!**

C.P. @ $x = 0$

$f(-2) = 0$

$f(0) = -4 \leftarrow$ abs min

$f(3) = \frac{1}{2} \leftarrow$ abs max

(b) Determine the intervals where the function is increasing and the intervals where it is decreasing.

f is increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$.

(c) Find the x-coordinate of each local extremum (local maximum and minimum).

There is only one local ~~max~~ min @ $x = 0$.

f': $-$ on the left of 0, $+$ on the right of 0

(d) Determine the intervals where the function is concave up and the intervals where it is concave down.

$f''(x) = \dfrac{10(1-3x^2)}{(x^2+1)^3}$.

$-\quad -\sqrt{\tfrac{1}{3}} \quad + \quad \sqrt{\tfrac{1}{3}} \quad -$

C.D. on $(-\infty, -\sqrt{\tfrac{1}{3}}) \cup (\sqrt{\tfrac{1}{3}}, \infty)$

C.U. on $(-\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}})$

(e) Determine the x-values where the inflection points occur.

$x = \sqrt{\tfrac{1}{3}}$ and $x = -\sqrt{\tfrac{1}{3}}$ are both in the domain of $f(x)$ and f'' changes sign at both, so they are both I.P.s of f.

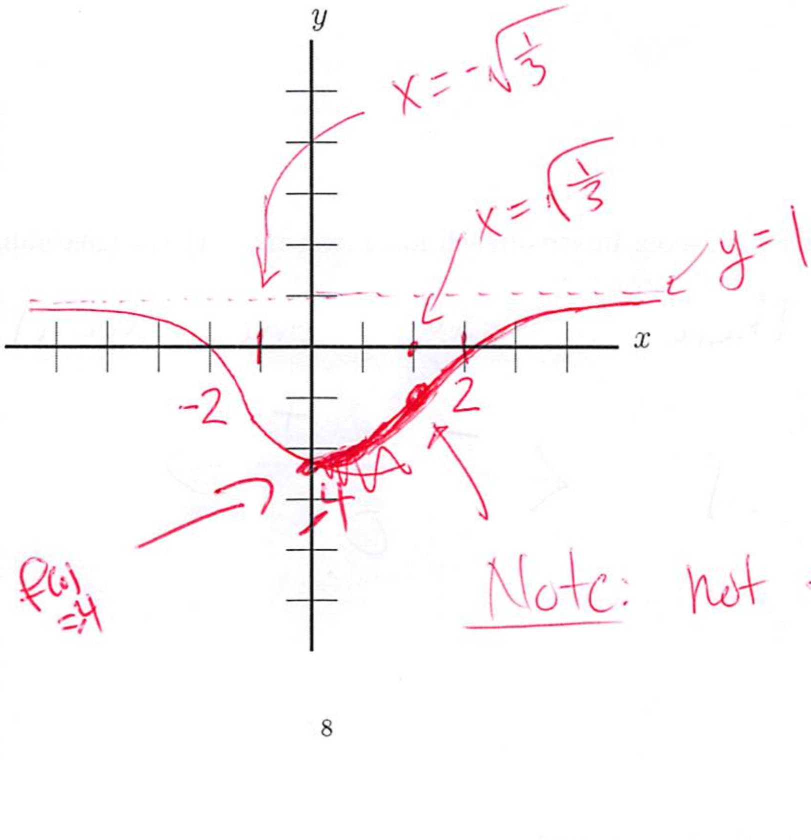
(f) Determine all vertical and horizontal asymptotes.

$\lim_{x \to \pm\infty} \dfrac{x^2-4}{x^2+10} = 1$

$\therefore$ H.A. of $y = 1$.

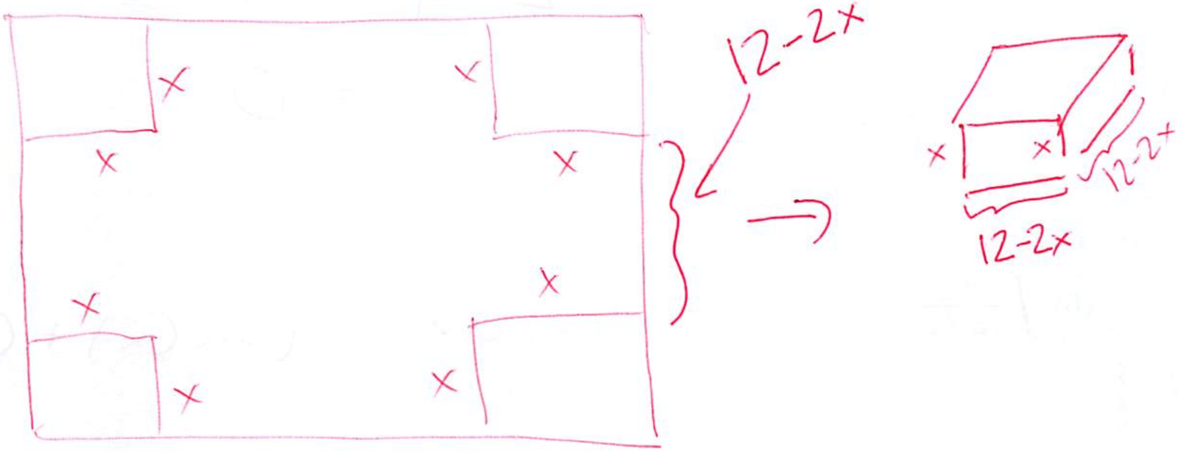
(no vert. asymptotes)

(g) Sketch the graph of $y = f(x)$.



Note: not to scale!

Problem 8 (10 points). A box with no top is constructed by cutting equal-sized squares from the corners of a 12cm $\times$ 12 cm sheet of metal and bending up the sides. What is the largest possible volume of such a box?



$$V = x(12-2x)(12-2x) \qquad \underline{[0,6]}$$

$$= x(144 - 48x + 4x^2)$$

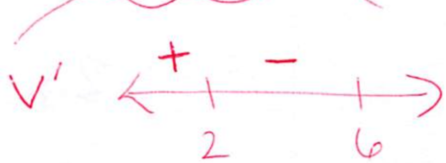
$$= 144x - 48x^2 + 4x^3$$

$$V' = 144 - 96x + 12x^2$$

$$= 12(12 - 8x + x^2)$$

$$= 12(x-2)(x-6).$$

also says it's a max



$V(0) = 0$ and $V(6) = 0$, and $V(2) > 0$ (so it's a max)

$$V(2) = 2(8)\cdot(8) = 128$$

Problem 9 (20 points). Evaluate the following integrals. **Show your work!**

(a) $\int_{-1}^{1} (x^3 - x)\, dx$ $= \frac{x^4}{4} - \frac{x^2}{2} \Big|_{-1}^{1} = \frac{1^4}{4} - \frac{1^2}{2} - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)$

$= 0$

(b) $\int \frac{dx}{\cos^2 x}$ $= \int \sec^2(x)\, dx = \tan(x) + C$

(c) $\int_0^2 x\sqrt{9 - 2x^2}\, dx$ $u = 9 - 2x^2, \quad du = -4x\, dx$

$\hookrightarrow -\frac{1}{4} du = x\, dx$

$\hookrightarrow \frac{-1}{4} \int_9^1 \sqrt{u}\, du = \frac{-1}{4}\left(\frac{2}{3} u^{3/2} \Big|_9^1\right) = \frac{-1}{4} \cdot \frac{2}{3}\left(1^{3/2} - 9^{3/2}\right)$

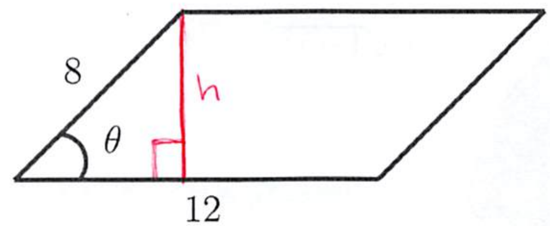
$= \frac{52}{12} = \frac{26}{6} = \frac{13}{3}$

(d) $\int \frac{2x^2 - 3x}{x}\, dx$

$= \int 2x - 3\, dx$

$= x^2 - 3x + C$

Problem 10 (10 points). A parallelogram has fixed side lengths 8cm and 12cm. The indicated angle θ is increasing at a rate of $\pi/4$ radians per second. How fast is the area changing when $\theta = \pi/3$ radians?



$$\sin(\theta) = \frac{h}{8}, \quad A = 12 \cdot h$$

$$\Rightarrow \quad \cos(\theta)\frac{d\theta}{dt} = \frac{1}{8}\frac{dh}{dt}$$

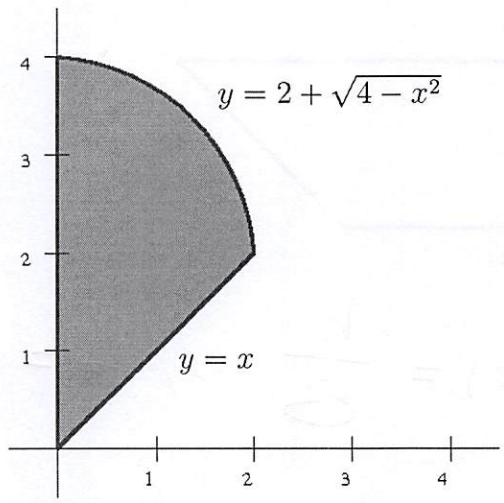
when $\theta = \pi/3$ and, $\frac{d\theta}{dt} = \pi/4$

$$\frac{dh}{dt} = \frac{1}{2}\cdot\frac{\pi}{4}\cdot 8 = \pi \text{ cm/sec.}$$

Now, $\frac{dA}{dt} = 12 \cdot \frac{dh}{dt}$

$$= 12\pi \text{ cm}^2\text{/sec}$$

Problem 11 (10 points). Consider the shaded region of the plane pictured below. It is bounded on the left by the y-axis, below by the line $y = x$, and above by the graph of $y = 2 + \sqrt{4 - x^2}$.



(a) Express the area of the shaded region using one or more *unevaluated* definite integrals.

$$\int_0^2 2 + \sqrt{4-x^2} - x \, dx$$

(b) Find the volume of the solid of revolution given by rotating the shaded region about the y-axis.

Shell Method!

$$2\pi \int_0^2 x\left(2 + \sqrt{4-x^2} - x\right) dx$$

(Use u-sub to evaluate this)

Problem 12 (10 points). Determine the values of the parameters a and b such that the following function $f(x)$ becomes continuous and differentiable at $x = 2$:

$$f(x) = \begin{cases} x^2 - 2x + b & \text{for } x > 2 \\ ax & \text{for } x \leq 2. \end{cases}$$

We want $\lim_{x \to 2} f(x)$ ~~to~~ to exist so

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 - 2x + b = 4 - 4 + b$$
$$= b$$

and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax = 2a$

so, $b = 2a$.

In a similar (but more complicated) way we want $\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x)$.

This means that $2(2) - 2 = a$

$$\boxed{2 = a}$$

And further, that $b = 4$.