

Name:

Section: 2 4 (circle one)

1. A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters the Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

Let $g(x, y, z) = 4x^2 + y^2 + 4z^2 - 16$. Now, $g=0$ is our constraint! By Lagrange, we look for points such that $\nabla T = \lambda \nabla g$, and then ~~we~~ check the values of T at those points. Here we go:

$$\nabla g = 8xi + 2yj + 8zk \quad \text{and} \quad \nabla T = 16xi + 4yzj + (4y - 16)k \quad \text{gives the}$$

following system of equations:

$$1) \quad 16x = 8\lambda x$$

From 1) we see that $x(16 - 8\lambda) = 0$, and thus,

$$2) \quad 8z = 2y\lambda$$

$$x=0 \quad \text{or} \quad \lambda=2.$$

$$3) \quad 4y - 16 = 8\lambda z$$

Case 1: $\lambda=2$; In this case, 2) gives us $z=y$ and this combined w/ 3) gives $4y - 16 = 16y$ and so, $y = -\frac{4}{3}$. Since $z=y$, $z = -\frac{4}{3}$. Using $g=0$,

$$4x^2 + \left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 = 16 \Rightarrow x^2 = 4 - \frac{4}{9} - \frac{16}{9} = \frac{16}{9} \Rightarrow x = \pm \frac{4}{3}.$$

This gives us the points $\left(\frac{-4}{3}, \frac{-4}{3}, \frac{-4}{3}\right)$ and $\left(\frac{4}{3}, \frac{-4}{3}, \frac{-4}{3}\right)$.

Case 2: $x=0$. Note that $g=0$ is now, $y^2 + 4z^2 = 16$. Using this and 2), we have $y^2 + \lambda^2 z^2 = 16$ and this gives $y = \pm \frac{4}{\sqrt{1+\lambda^2}}$. Using 3) one can obtain the equation

$$\frac{16}{\sqrt{1+\lambda^2}} - 16 = 8\lambda \left(\frac{4}{\sqrt{1+\lambda^2}} \frac{\lambda}{2}\right), \text{ from here we have } 1 - \lambda^2 - \sqrt{1+\lambda^2} = 0. \text{ Multiplying both sides by the conjugate gives } (1-\lambda^2)(1+\lambda^2) = 0 \text{ and whence } \lambda^2(\lambda^2-3) = 0.$$

Since $\lambda \neq 0$ (ever) we get back a bunch of points:

$$(0, \underbrace{\frac{4}{\sqrt{1+3}}}_{\frac{4}{2}}, \sqrt{3}), (0, -2, -\sqrt{3}), (0, 2, -\sqrt{3}), (0, -2, \sqrt{3})$$

2 (One checks that $\left(\frac{4}{3}, \frac{-4}{3}, \frac{-4}{3}\right)$ is the hot point)
(est)

2. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where the function $f(x, y, z) = x + 2y + z$ has its minimum and maximum values.

let $g = x^2 + y^2 + z^2 - 25$, Now... if $\nabla f = \lambda \nabla g$ we have

$$\begin{aligned} 1 &= 2\lambda x & x &= \frac{1}{2\lambda} \\ 2 &= 2\lambda y \Rightarrow y &= \frac{1}{\lambda} \\ 1 &= 2\lambda z & z &= \frac{1}{2z} \end{aligned}$$

Using $g = 0$, we have

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{2z}\right)^2 = 25$$

$$\Leftrightarrow \frac{6}{4\lambda^2} = 25 \Leftrightarrow \lambda^2 = \frac{3}{50}.$$

When $\lambda = +\sqrt{3/50}$ we have $\left(\frac{1}{2\sqrt{3/50}}, \frac{1}{\sqrt{3/50}}, \frac{1}{2\sqrt{3/50}}\right)$.

When $\lambda = -\sqrt{3/50}$ we have $\left(\frac{-1}{2\sqrt{3/50}}, \frac{-1}{\sqrt{3/50}}, \frac{-1}{\sqrt{3/50}}\right)$.

So, we have a max at $\lambda = +\sqrt{3/50}$ and min at $\lambda = -\sqrt{3/50}$.

(after plugging these values into $f(x, y, z)$...)

3. Find the absolute minimum and maximum of the function $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular region $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

4. Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

Let $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$

and $g = x + 2y + 3z - 13$. If $\nabla f = \lambda \nabla g$, then

$$2x-2 = \lambda, \quad 2y-2 = 2\lambda, \quad 2z-2 = 3\lambda$$

$$1) x = \frac{\lambda}{2} + 1 \quad \underbrace{y-1 = \lambda}_{y=1+\lambda} \quad \frac{2}{3}z - \frac{2}{3} = \lambda$$

$$2) y = \lambda + 1 \quad \Rightarrow z = \frac{3}{2}\lambda + 1$$

Using $g=0$ we have $\frac{1}{2} + 1 + 2\lambda + 2 + \frac{9}{2}\lambda + 3 = 13$

$$7\lambda = 7 \Rightarrow \lambda = 1$$

$$\Rightarrow \left(\frac{3}{2}, 2, \frac{5}{2} \right)$$

5. For the following functions, use Taylor series to find a polynomial of degree 3 that approximates the function for points (x, y) "near" the origin.

$$f(x, y) = \ln(3x + 4y + 1)$$

$$f_x = \frac{1}{3x+4y+1} \cdot 3, \quad f_y = \frac{1}{3x+4y+1} \cdot 4, \quad f_{xy} = \frac{-12}{(3x+4y+1)^2}, \quad f_{x^2} = \frac{-9}{(3x+4y+1)^2}$$

$$f_{y^2} = \frac{-16}{(3x+4y+1)^2}, \quad f_{x^2y} = \frac{72}{(3x+4y+1)^3}, \quad f_{xy^2} = \frac{96}{(3x+4y+1)^3}, \quad f_{x^3} = \frac{54}{(3x+4y+1)^2}$$

and $f_{y^3} = \frac{128}{(3x+4y+1)^3}$. So, the Taylor polynomial (of deg = 3) is

$$\begin{aligned} f(x, y) \approx & 3x + 4y - 12xy - \frac{9}{2}x^2 - \frac{16}{2}y^2 + \frac{72}{2}x^2y \\ & + \frac{96}{2}xy^2 + \frac{54}{3!}x^3 + \frac{128}{3!}y^3 \end{aligned}$$

$$f(x, y) = e^{x^2+y^2}$$

$$f_x = e^{x^2+y^2} \cdot 2x, \quad f_y = e^{x^2+y^2} \cdot 2y, \quad f_{xy} = e^{x^2+y^2} \cdot 4xy, \quad f_{xx} = 2e^{x^2+y^2} + e^{x^2+y^2} \cdot 4x^2$$

$$f_{yy} = 2e^{x^2+y^2} + e^{x^2+y^2} \cdot 4y^2, \quad f_{x^2y} = 2e^{x^2+y^2}(2y) + e^{x^2+y^2} \cdot 8yx^2,$$

$$f_{y^2x} = 4x e^{x^2+y^2} + e^{x^2+y^2} \cdot 8xy^2, \quad f_{x^3} = 4x \cdot 4x e^{x^2+y^2} + \frac{8x e^{x^2+y^2} + e^{x^2+y^2}}{8x^3}$$

$$f_{y^3} = 4y e^{x^2+y^2} + 8y e^{x^2+y^2} + e^{x^2+y^2} \cdot 8y^3, \quad \underline{\text{SC}}$$

$$f(x, y) \approx 1 + \frac{2}{2!}x^2 + \frac{2}{2!}y^2$$