

Name: Key

Section: 2 4 (circle one)

1. Find the distance between the point $(1, 3, 5)$ and the plane $x - y - z = 1$.

The point $\mathbf{Q} = (1, 0, 0)$ is on the plane (it satisfies the equation). Our normal vector is $\vec{n} = \langle 1, -1, -1 \rangle$. $\overrightarrow{PQ} = \langle 1-1, 0-3, 0-5 \rangle = \langle 0, -3, -5 \rangle$

$$d = |\text{proj}_{\vec{n}} \overrightarrow{PQ}| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{|\vec{n}|} = \frac{|1 \cdot 0 + (-1) \cdot (-3) + (-1) \cdot (-5)|}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{8}{\sqrt{3}}$$

2. Given the parametric equations $x = \cos^2(t)$ and $y = \sin^2(t)$, find an equation for the tangent line when $t = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin(t)\cos(t)}{2\cos(t)(-\sin(t))}, \quad \left. \frac{dy}{dt} \right|_{t=\frac{\pi}{4}} = \frac{2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}}{-2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}} = -1$$

$$x(\frac{\pi}{4}) = (\cos(\frac{\pi}{4}))^2 = (\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$$

$$y(\frac{\pi}{4}) = (\sin(\frac{\pi}{4}))^2 = (\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$$

so, the equation of the tangent line is $y - \frac{1}{2} = -1(x - \frac{1}{2})$

3. Find the length of the curve given by $x = \frac{y^4}{4} + \frac{1}{8y^2}$ from $y = 1$ to $y = 2$.

$$L = \int_1^2 \sqrt{1 + (\frac{dy}{dx})^2} \, dy$$

$$= \int_1^2 \sqrt{1 + t^3 - \frac{1}{2} + \frac{1}{16t^6}} \, dt$$

$$= \int_1^2 \sqrt{(t^3 + \frac{1}{4t^3})^2} \, dt$$

$$= \int_1^2 t^3 + \frac{1}{4t^3} \, dt = \frac{t^4}{4} + \frac{1}{-8t^2} \Big|_{t=1}^{t=2} = \frac{2^4}{4} - \frac{1}{8 \cdot 2^2} - \left(\frac{1^4}{4} - \frac{1}{8 \cdot 1^2} \right) \\ = 4 - \frac{1}{32} - \frac{1}{4} + \frac{1}{8} \\ = \frac{123}{32}$$

4. Graph $r = 4\cos(\theta)$ in polar coordinates. Give the equation of the tangent line when $\theta = \frac{\pi}{6}$.

$$f'(\theta) = \frac{dr}{d\theta} = -4\sin(\theta)$$

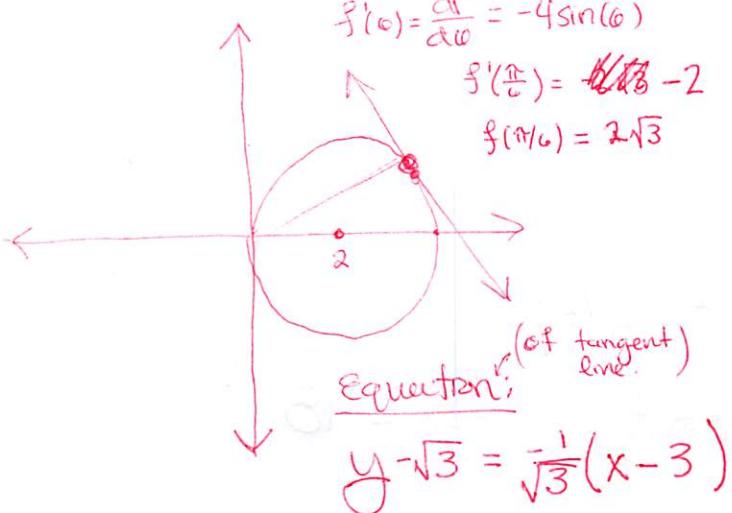
$$f'(\frac{\pi}{6}) = -4\sin(\frac{\pi}{6}) = -2$$

$$f(\frac{\pi}{6}) = 2\sqrt{3}$$

$$x = r\cos(\theta), \rightarrow x = 4\cos(\theta) \cdot \cos(\theta)$$

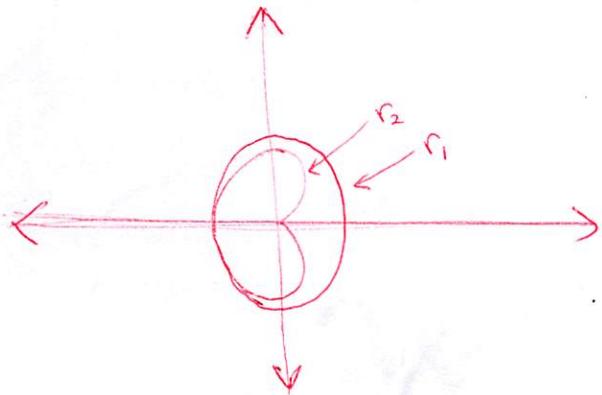
$$y = r\sin(\theta), \rightarrow y = 4\cos(\theta) \cdot \sin(\theta)$$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$



$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{6}} = \frac{2\sqrt{3}/2}{-2 \cdot \frac{\sqrt{3}}{2} + 2\sqrt{3} \cdot \frac{1}{2}} = \frac{-1}{\sqrt{3}}$$

5. Find the area between the curves $r_2 = 1 - \cos(\theta)$ and $r_1 = 2$.



$$\begin{aligned}
 \text{Area: } & \frac{1}{2} \int_0^{2\pi} 2^2 - (1 - \cos(\phi))^2 d\phi \\
 &= \frac{1}{2} \int_0^{2\pi} 4 - (1 - 2\cos(\phi) + \cos^2(\phi)) d\phi \\
 &= \frac{1}{2} \int_0^{2\pi} 3 + 2\cos(\phi) - \cos^2(\phi) d\phi \\
 &= \frac{1}{2} \int_0^{2\pi} 3 + 2\cos(\phi) - \left(\frac{1 + \cos(2\phi)}{2} \right) d\phi \\
 &= \frac{1}{2} \int_0^{2\pi} \frac{5}{2} + 2\cos(\phi) - \frac{\cos(2\phi)}{2} d\phi \\
 &= \frac{1}{2} \left(\frac{5\phi}{2} + 2\sin(\phi) - \frac{\sin(2\phi)}{4} \right) \Big|_{\phi=0}^{\phi=2\pi} \\
 &= \frac{5\pi}{2}
 \end{aligned}$$

6. Find the length of the spiral $r = \theta^2$ from $0 \leq \theta \leq \sqrt{12}$.

$$\begin{aligned}
 L &= \int_0^{\sqrt{12}} \sqrt{(\phi^2)^2 + (2(\phi))^2} d\phi \\
 &= \int_0^{\sqrt{12}} \sqrt{\phi^4 + 4\phi^2} d\phi \\
 &= \int_0^{\sqrt{12}} \sqrt{\phi^2 + 4} \phi d\phi \quad u = \phi^2 + 4, \quad du = 2\phi d\phi \\
 &\quad \Rightarrow \frac{1}{2} du = \phi d\phi \\
 &= \int_4^{16} \sqrt{u} \left(\frac{1}{2} du \right) \\
 &= \frac{1}{2} \frac{u^{3/2}}{3/2} \Big|_4^{16} = \frac{1}{3} (16^{3/2} - 4^{3/2}) \\
 &= \frac{1}{3} (64 - 8) = \frac{54}{3} = 18
 \end{aligned}$$