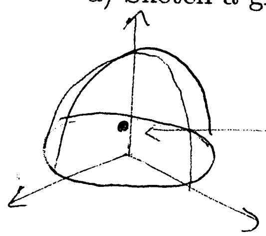
**Worksheet 8** / 15 points

Name: Solutions

Section: 2 4 (circle one)

1. Consider the function $f(x, y) = \sqrt{1 - x^2 - y^2}$.

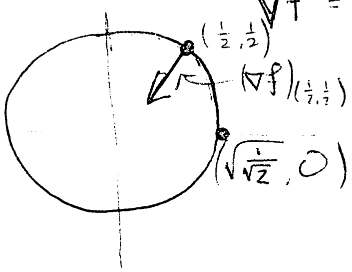
a) Sketch a graph of this function and plot the point $(\frac{1}{2}, \frac{1}{2}, f(\frac{1}{2}, \frac{1}{2}))$.



$f(\frac{1}{2},\frac{1}{2}) = \sqrt{1-\frac{1}{4}-\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

This point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$

b) Find $\nabla f$. Plot the level curve of $f$ containing the point $(\frac{1}{2}, \frac{1}{2})$ and plot the vector $(\nabla f)_{(\frac{1}{2},\frac{1}{2})}$ at this point.




$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j = \frac{1}{2\sqrt{1-x^2-y^2}}(-2x)\, i + \frac{1}{2\sqrt{1-x^2-y^2}}(-2y)\, j$

$= \frac{-x}{\sqrt{1-x^2-y^2}} i + \frac{-y}{\sqrt{1-x^2-y^2}} j$

$(\nabla f)_{(\frac{1}{2},\frac{1}{2})} = \frac{-1}{\sqrt{2}} i + \frac{-1}{\sqrt{2}} j$

c) Find the derivative of $f$ in the direction $u = 2i + 3j$ at the point $(0, \frac{1}{\sqrt{2}})$

$(D_u f)_{(0,\frac{1}{\sqrt{2}})} = \left(\nabla f \cdot \frac{u}{|u|}\right)_{(0,\frac{1}{\sqrt{2}})} = -j \cdot \frac{2i+3j}{\sqrt{13}}$

$= -\frac{3}{\sqrt{13}}$

Note: $(\nabla f)_{(0,\frac{1}{\sqrt{2}})} = \frac{-1/\sqrt{2}}{\sqrt{1-\frac{1}{2}}} j$

$= -j$

d) Find the direction in which $f$ is increasing most rapidly. Find the derivative of $f$ in this direction.

In general, $D_{\nabla f} f = \nabla f \cdot \frac{\nabla f}{|\nabla f|}$

$= |\nabla f|$

$= \sqrt{\frac{+x^2}{1-x^2-y^2} + \frac{+y^2}{1-x^2-y^2}} = \sqrt{\frac{x^2+y^2}{1-x^2-y^2}}$

As we know, $\nabla f$ points in the direction $f$ increases most rapidly.

e) Find the equation of the tangent plane to the surface at $(\frac{1}{2}, \frac{1}{2}, f(\frac{1}{2}, \frac{1}{2}))$

$(\nabla f)_{(\frac{1}{2},\frac{1}{2})} = \frac{-1}{\sqrt{2}} i + \frac{-1}{\sqrt{2}} j$, Tangent Plane: $\frac{-1}{\sqrt{2}}(x-\frac{1}{2}) - \frac{1}{\sqrt{2}}(y-\frac{1}{2}) - (z-\frac{1}{\sqrt{2}}) = 0$

f) Find the equations of the normal line to the plane at this point.

$x = \frac{1}{2} - \frac{1}{\sqrt{2}}t$, $y = \frac{1}{2} - \frac{1}{\sqrt{2}}t$, $z = \frac{1}{\sqrt{2}} - t$

2. Consider the function $f(x, y, z) = xy + xz + zy + xyz$.

a) Show that the point $(1, 1, 1)$ is on the level surface $4 = f(x, y, z)$.

Since $f(1,1,1) = 1 + 1 + 1 + 1 = 4$, $(1,1,1)$ is on said surface.

b) Find the equation of the tangent plane to the level surface at this point.

$$\nabla f = (y + z + yz)i + (x + z + xz)j + (x + y + xy)k$$

$(\nabla f)_{(1,1,1)} = 3i + 3j + 3k$ $\therefore$ $3(x-1) + 3(y-1) + 3(z-1) = 0$ is the equation of the tangent plane.

c) Find the equation of the normal line at this point.

$x = 1 + 3t$, $y = 1 + 3t$, $z = 1 + 3t$

3. Consider the curve given by the intersection of the surfaces

$$14 = x^2 + y^2 + z^2 \quad \text{and} \quad 0 = x^2 - y^2 - z.$$

a) Show that $(2, 1, 3)$ is in the intersection of both the surfaces.

Since $14 = 2^2 + 1^2 + 3^2$ and $0 = 2^2 - 1^2 - 3$, $(2,1,3)$ is on both surfaces!

b) Give the equation of the tangent line to the curve of intersection at the point $(2, 1, 3)$.

let $f(x,y,z) = x^2 + y^2 + z^2 - 14$ and $g(x,y,z) = x^2 - y^2 - z$.

$\nabla f = 2x\,i + 2y\,j + 2z$ $\qquad$ $\nabla g = 2x\,i - 2y\,j - k$

$$(\nabla f \times \nabla g)_{(2,1,3)} = \begin{vmatrix} i & j & k \\ 4 & 2 & 6 \\ 4 & -2 & -1 \end{vmatrix} = (\cancel{\quad})i - (-4-24)j + (-8-8)k$$

$(-2+12)$

$= 10i - 28j - 16k$

$\therefore$ $x = 2 + 10t$, $y = 1 - 28t$, $z = 3 - 16t$ give parametric equations of the tangent line @ $(2,1,3)$.