Math 242 Final

Solutions by Kenny

Name: ____________________

Please circle your section:

Recitation 1	Thurs 12-12:50	TA - Dan Flores
Recitation 2	Thurs 1:30-2:20	TA - Dan Flores
Recitation 3	Tues 9-9:50	TA - Vince Chung
Recitation 4	Tues 12-12:50	TA - Vince Chung
Recitation 5	Wed 9:30-10:20	TA - Lance Ferrer
Recitation 6	Wed 12:30-1:20	TA - Lance Ferrer
Recitation 7	Fri 10:30-11:20	TA - Ikenna Nometa
Recitation 8	Fri 12:30-1:20	TA - Ikenna Nometa
Recitation 9	Fri 9:30-10:20	TA - Dan Flores

Question	Points	Score
1	12	
2	10	
3	12	
4	10	
5	13	
6	12	
7	15	
8	15	
9	13	
10	6	
11	17	
12	10	
13	5	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. Find the derivative of each of the following functions.

(a) (5 points) $f(x) = 2\sin^{-1}(x^2)$

$$f'(x) = 2 \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

(b) (7 points) $g(x) = 2^{x+\ln(x)}$

$$g'(x) = 2^{x+\ln(x)} \cdot \ln 2 \cdot \left(1 + \frac{1}{x}\right)$$

2. (10 points) Evaluate the following integral: $\int x \sec^2 x \, dx$ Parts

$u = x \qquad dv = \sec^2 x \, dx$

$du = dx \qquad v = \tan x$

$= x \tan x - \int \tan x \, dx$

$= x \tan x - \ln|\sec x| + c$

3. Evaluate the following integrals.

(a) (7 points) $\int_0^{\pi/6} \cos^3(3x)\sin^2(3x)\,dx$

ODD power cos use

$u = \sin 3x$

$du = 3\cos 3x\,dx$

$\cos^2 3x = 1 - \sin^2 3x$

$= 1 - u^2$

$u(0) = 0$

$u(\pi/6) = 1$

$$= \int_0^{\pi/6} \cos^2(3x)\sin^2(3x)\cos(3x)\,dx$$

$$= \int_0^1 (1-u^2)\,u^2\left(\tfrac{1}{3}du\right)$$

$$= \frac{1}{3}\int_0^1 u^2 - u^4\,du$$

$$= \frac{1}{3}\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{3} - \frac{1}{5}\right)$$

(b) (5 points) $\int \frac{e^x}{e^x+1}\,dx \quad = \int \frac{du}{u}$

$u = e^x + 1$

$du = e^x\,dx$

$$= \ln|u| + C$$

$$= \ln|e^x + 1| + C$$

4. (10 points) Evaluate the following integral. (Hint: Use trigonometric substitution.)

$$\int \frac{dx}{(9-x^2)^{3/2}}$$

pattern $9-x^2$ use $\sin$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= \int \frac{3\cos\theta\, d\theta}{27\cos^3\theta}$$

$$(9-x^2)^{3/2} = (9 - 9\sin^2\theta)^{3/2}$$

$$= (9\cos^2\theta)^{3/2}$$

$$= (3\cos\theta)^3$$

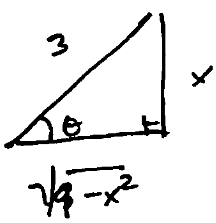
$$= 27\cos^3\theta$$

$$= \frac{1}{9}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{9}\tan\theta + C$$

$$= \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C$$

$$\sin\theta = \frac{opp}{hyp} = \frac{x}{3}$$



5. Determine whether the following improper integrals converge or diverge, and evaluate those that converge.

(a) (5 points) $\int_1^3 \frac{2}{\sqrt{x-1}}\,dx$ not defined at 1.

$$= \lim_{t \to 1^+} \int_t^3 \frac{2}{\sqrt{x-1}}\,dx \qquad \begin{array}{ll} u = x-1 & u(3) = 2 \\ du = dx & u(t) = t-1 \end{array}$$

$$= \lim_{t \to 1^+} \int_{t-1}^{2} \frac{2}{u^{1/2}}\,du = \lim_{t \to 1^+} \int_{t-1}^{2} 2u^{-1/2}\,du$$

$$= \lim_{t \to 1^+} 4u^{1/2} \Big|_{t-1}^{2} = \lim_{t \to 1^+} \left[4\sqrt{2} - 4\sqrt{t-1}\right] = 4\sqrt{2}$$

(b) (5 points) $\int_3^\infty \frac{2}{\sqrt{x-1}}\,dx$ comparison test

$$\frac{2}{\sqrt{x-1}} > \frac{2}{\sqrt{x}}$$

and $\int_3^\infty \frac{2}{\sqrt{x}}\,dx$ diverges (p-integral $p = 1/2$)

So $\int_3^\infty \frac{2}{\sqrt{x-1}}$ diverges

(c) (3 points) What does your answer to (b) tell you about the series $\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-1}}$?

Since $f(x) = \frac{2}{\sqrt{x-1}}$ is cts, pos, dec, & interpolates

then $\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-1}}$ diverges.

6. Evaluate the following limits. If a limit does not exist write DOES NOT EXIST.

(a) (4 points) $\lim_{n\to\infty} (-1)^n \frac{n}{n+1}$ Doesn't exist.

odd terms $\longrightarrow -1$

even terms $\longrightarrow +1$

(b) (4 points) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \quad \text{geometric}$$

$$= \frac{(1/3)^1}{1 - 1/3} \qquad \frac{\text{first term}}{1-\text{ratio}}$$

$$= \frac{1}{2}$$

(c) (4 points) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n$ $= e^2$ common limit.

7. For each of the following series decide if it converges or diverges and explain why.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p=2$), then

$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges.

(b) (5 points) $\sum_{n=1}^{\infty} \frac{2^n}{n \cdot n!}$ Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{(n+1)(n+1)!} \cdot \frac{n \cdot n!}{2^n}\right| = 2 \cdot \frac{n}{(n+1)^2} \to 0 < 1$$

series converges

(c) (5 points) $\sum_{n=3}^{\infty} (-1)^n \frac{\ln(n)}{n}$ AST.

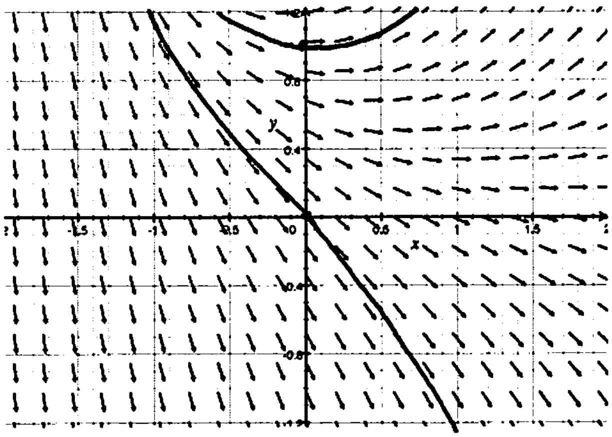
$$b_n = \frac{\ln(n)}{n}$$

(i) b_n's are positive & decreasing (n beats $\ln n$)

(ii) $\lim_{n \to \infty} \frac{\ln(n)}{n} \overset{L}{=} \lim_{n \to \infty} \frac{1}{n} = 0.$

$\therefore$ series converges

8. Consider the differential equation $y' = y - e^{-x}$, with slope field pictured below.



$$y' - y = -e^{-x}$$

(a) (4 points) Sketch (on the slope field) the solutions satisfying $y(0) = 0$ and $y(0) = 1$.

(b) (7 points) Find the general solution of the differential equation.

$P(x) = -1$

Integrating factor:

$$I(x) = e^{\int P(x)dx} = e^{\int -1\,dx} = e^{-x}$$

$$e^{-x}y' - e^{-x}y = -e^{-2x}$$

$$(e^{-x}y)' = -e^{-2x}$$

$$e^{-x}y = \int -e^{-2x}\,dx$$

$$e^{-x}y = \frac{1}{2}e^{-2x} + c$$

$$y = e^{x}\left(\frac{1}{2}e^{-2x} + c\right) = ce^{x} + \frac{1}{2}e^{-x}$$

(c) (4 points) Find the particular solution satisfying $y(0) = 0$.

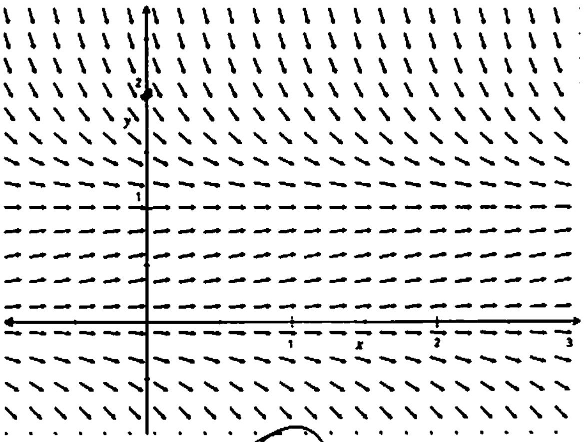
$$0 = y(0) = c \cdot 1 + \frac{1}{2}$$

$$\Rightarrow c = -\frac{1}{2}$$

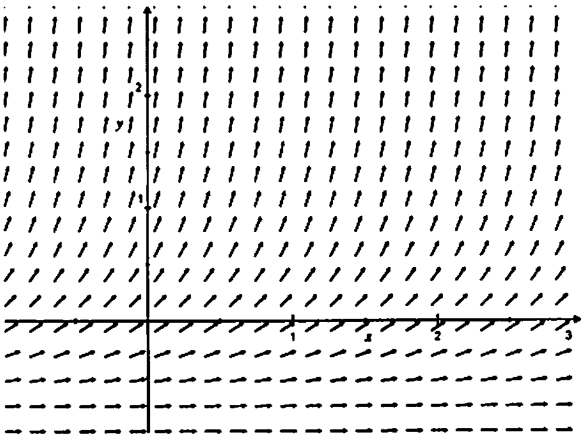
$$y = ce^{x} + \frac{1}{2}e^{-x} = -\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}.$$

9. Consider the differential equation: $y' = y(1-y)$.

(a) (4 points) Which of the following plots represents the direction field of this differential equation? Circle your answer.



(I)



(II)

(b) (2 points) If y is the solution satisfying $y(0) = 2$, what is $\lim_{x\to\infty} y$?
(Hint: You can read this directly off the slope field.)

1

(c) (7 points) Solve the differential equation.

$$\frac{dy}{dx} = y(1-y)$$

$$\frac{1}{y(1-y)}\,dy = dx$$

$$\int \frac{1}{y(1-y)}\,dy = \int dx$$

$$\int \frac{1}{y} + \frac{1}{1-y} = \int dx$$

$$\ln|y| - \ln|1-y| = x + c$$

$$\ln\left|\frac{y}{1-y}\right| = x + c$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

$$1 = A(1-y) + By$$

$y=1$: $1 = A\cdot 0 + B\cdot 1 \Rightarrow B = 1$

$y=0$: $1 = A\cdot 1 + B\cdot 0 \Rightarrow A = 1.$

$$\left|\frac{y}{1-y}\right| = e^{x+c} \quad (y=0 \text{ is a soln})$$

$$\Rightarrow \frac{y}{1-y} = Ae^x, \quad A \in \mathbb{R}.$$

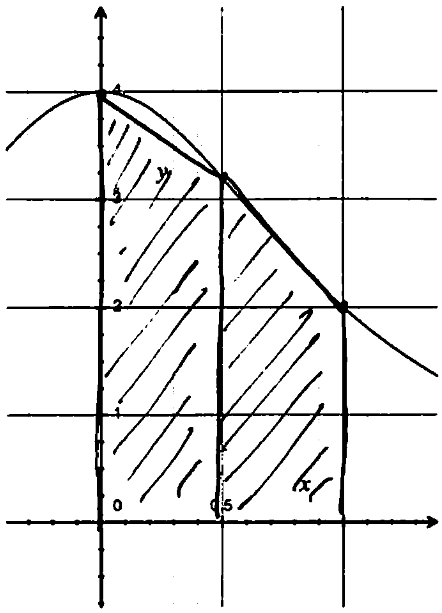
$$y = Ae^x - Ae^x y$$

$$(1 + Ae^x)y = Ae^x$$

$$y = \frac{Ae^x}{1+Ae^x}$$

10. In this problem, you will use numerical integration to estimate $\pi = \int_0^1 \frac{4\,dx}{1+x^2}$.

(a) (2 points) The graph the function $y = 4/(1+x^2)$ between $x = 0$ and $x = 1$ is shown below. On the graph draw, and shade in, the trapezoids whose area is computed by the Trapezoidal Rule with $n = 2$.



(b) (4 points) Use the Trapezoidal Rule with $n = 2$ to estimate the integral $\int_0^1 \frac{4\,dx}{1+x^2}$. Your answer should be a fraction (or decimal number).

$a = 0, \quad b = 1, \quad n = 2, \quad \Delta x = \frac{b-a}{n} = \frac{1}{2}$

x_i	0	$1/2$	1
$y_i = \frac{4}{1+x_i^2}$	4	$\frac{4}{1+1/4} = \frac{16}{5}$	2
T_2 coeffs	1	2	1
	4	$\frac{32}{5}$	2

mult.

$$T_2 = \frac{\Delta x}{2} \cdot \text{Sum} = \frac{1/2}{2}\left(4 + \frac{32}{5} + 2\right)$$

11. Consider the power series: $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n \cdot 2^n}$

(a) (13 points) Find its interval of convergence. (Hint: Check the endpoints.)

Root test:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left| \frac{(-1)^n (x-2)^n}{n \cdot 2^n} \right|} = \frac{|x-2|}{2} \cdot \frac{1}{\sqrt[n]{n}} \longrightarrow \frac{|x-2|}{2}$$

$$\frac{|x-2|}{2} < 1 \iff |x-2| < 2 \iff -2 < x-2 < 2$$
$$\iff 0 < x < 4$$

Endpt $x = 0$: $\sum_{n=1}^{\infty} (-1)^n \frac{(0-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(p-series $p \geq 1$).

Endpt $x=4$: $\sum_{n=1}^{\infty} (-1)^n \frac{(4-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ doesn't converge

absolutely (see above), but converges conditionally by AST ($b_n = \frac{1}{n}$ is pos, dec & $b_n \to 0$).

I.O.C $= (0, 4]$

(b) (2 points) What is its radius of convergence?

$R = 2$

(c) (2 points) For which values of x does the series converge absolutely?

$(0, 4)$

12. Consider the function $f(x) = x\cos(3x)$.

(a) (5 points) Write down the Taylor series for $f(x)$ based at $a = 0$. (Hint: Manipulate a 'famous Maclaurin series' – do not calculate derivatives.)

$$x\cos(3x) = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n} \cdot x^{2n} \cdot x}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{9^n \cdot x^{2n+1}}{(2n)!}$$

(b) (5 points) Use your result in (a) to find $f^{(5)}(0)$ (the fifth derivative of f).

The coefficient on the x^5 term is

$$\frac{f^{(5)}(0)}{5!} = \frac{(-1)^2 \cdot 9^2}{4!} \quad \text{(power is 5 when } n=2)$$

$$\Rightarrow f^{(5)}(0) = 5 \cdot 81 = 405$$

13. (5 points) The degree 3 Taylor polynomial centered at $a = 0$ for the function $\sin(x/2)$ is

$$\sin(x/2) \approx \frac{x}{2} - \frac{x^3}{8 \cdot 3!}. \tag{1}$$

Estimate the error in this approximation when $|x| < 0.1$.

$f(x) = \sin(x/2)$

$f'(x) = 2\cos(x/2)$

$f''(x) = 4\sin(x/2)$

$f'''(x) = -8\cos(x/2)$

$f^{(4)}(x) = 16 \cdot \sin(x/2)$

$|f^{(4)}(x)| = |16 \cdot \sin x/2| \le 16$

we may use $M = 16$.

so

$$|R_4(x)| \le \frac{M \cdot |x-a|^5}{5!} = \frac{16 \cdot |x|^5}{5!}$$

$$\le \frac{16 \cdot (0.1)^5}{5!}$$

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$
$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$
$$\sin x \cos x = \frac{1}{2}\sin(2x)$$
$$\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$
$$\cos x \cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$
$$\sin x \cos y = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

- Integrals of trigonometric functions.

$$\int \tan x \; dx = \ln|\sec x| + C$$
$$\int \cot x \; dx = \ln|\sin x| + C$$
$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x \; dx = -\ln|\csc x + \cot x| + C$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.