Math 242 Final Spring 2016

Name: ______________________ **Section:** ______

Instructor: ______________________

Solutions
by D. Yuen

Question	Points	Score
1	21	
2	11	
3	8	
4	12	
5	13	
6	10	
7	9	
8	5	
9	5	
10	6	
Total:	100	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. For each of the following definite and indefinite integrals, evaluate it or show that it diverges.

(a) (6 points) $\int_0^{\pi/2} x\cos(x)dx$ <u>Integration by parts</u>

$u = x \quad dv = \cos x \, dx$

$du = dx \quad v = \sin x$

Indefinite integral first:

$x \sin x - \int \sin x \, dx$

$x \sin x - -\cos x + C$

$= x \sin x + \cos x \Big|_0^{\pi/2}$

$= \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right) - (0 \sin 0 + \cos 0)$

$= \frac{\pi}{2} \cdot 1 + 0 - (0 + 1)$

$\boxed{= \frac{\pi}{2} - 1}$

(b) (7 points) $\int \frac{x^2}{(4-x^2)^{3/2}}dx$

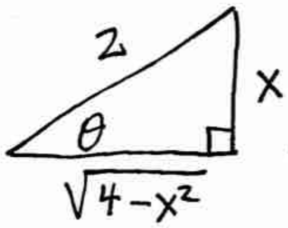
Trig sub.

Patter $\sqrt{a^2-x^2}$.

$$= \int \frac{x^2 dx}{(\sqrt{4-x^2})^3}$$

Let $x = 2\sin\theta \Rightarrow \frac{x}{2} = \sin\theta$

so that $\sqrt{2^2-x^2} = 2\cos\theta$



$dx = 2\cos\theta\, d\theta$

$$= \int \frac{(2\sin\theta)^2\, 2\cos\theta\, d\theta}{(2\cos\theta)^3}$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} d\theta = \int \tan^2\theta\, d\theta$$

$\tan\theta = \frac{x}{\sqrt{4-x^2}}$

$$= \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C$$

(c) (8 points) $\int_3^\infty \frac{1}{x(2x-1)}dx$

Indefinite integral first. Partial fractions

$$\frac{1}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$$1 = A(2x-1) + Bx$$

$$x=0 \Rightarrow \quad 1 = A(-1)+0 \Rightarrow -1 = A$$

$$x=\tfrac{1}{2} \Rightarrow \quad 1 = 0 + B(\tfrac{1}{2}) \Rightarrow 2 = B$$

$$\int\left(\frac{-1}{x} + \frac{2}{2x-1}\right)dx = -\ln|x| + \frac{2\ln|2x-1|}{2} + C$$

$$= -\ln|x| + \ln|2x-1| + C$$

Improper integral

$$= \lim_{b\to\infty}\int_3^b \frac{1}{x(2x-1)}dx = \lim_{b\to\infty}\Big[-\ln|x| + \ln|2x-1|\Big]_3^b$$

$$= \lim_{b\to\infty}\left[-\ln b + \ln(2b-1) - (-\ln 3 + \ln 5)\right]$$

Type $-\infty+\infty$

Combine into one ln

$$= \lim_{b\to\infty}\left[\ln\left(\frac{2b-1}{b}\right) + \ln 3 - \ln 5\right]$$

$$= \lim_{b\to\infty}\left[\ln\left(2-\frac{1}{b}\right) + \ln 3 - \ln 5\right]$$

$$= \ln(2-0) + \ln 3 - \ln 5$$

$$= \ln 2 + \ln 3 - \ln 5$$

2. For each of the following series decide if it converges or diverges and explain why.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{n^4+n^2}{n^5+n}$ ⇦ Behaves like $\sum \frac{n^4}{n^5} = \sum \frac{1}{n}$ so compare to $\sum \frac{1}{n}$.

positive series

Limit compare with $\sum \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{n^4+n^2}{n^5+n} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n^5+n^3}{n^5+n} \cdot \frac{\frac{1}{n^5}}{\frac{1}{n^5}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1+\frac{1}{n^4}} = \frac{1+0}{1+0} = 1 \neq 0.$$

Since $\sum \frac{1}{n}$ diverges (p-series with $p=1 \leq 1$)

then $\sum_{n=1}^{\infty} \frac{n^4+n^2}{n^5+n}$ diverges

(b) (6 points) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ ⇦ Note $\sin(n)$ is +/− and not alternating. So the only hope is absolute convergence.

Consider $\sum \left| \frac{\sin(n)}{n^2} \right|$.

Note $\frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$ all n.

Since $\sum \frac{1}{n^2}$ converges (p-series with $p=2>1$)

then $\sum \frac{|\sin(n)|}{n^2}$ converges by regular comparison.

Then $\sum \frac{\sin(n)}{n^2}$ converges absolutely.

3. For each of the following series, determine its sum.

(a) (4 points) $\sum_{n=0}^{\infty}(-1)^n \frac{1}{4^n}$

Geometric: ratio $= -\frac{1}{4}$

first term $= \frac{(-1)^0}{4^0} = 1$.

$$\text{Sum} = \frac{1}{1 - \frac{-1}{4}} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$$

(b) (4 points) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

This resembles $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

In fact, it is just with $x = -1$.

$$\text{sum} = e^{-1}$$

4. Find the derivative of each of the following functions.

(a) (6 points) $f(x) = 2^x \ln(x)$

product rule.

$$f'(x) = 2^x \ln 2 \cdot \ln x + 2^x \cdot \frac{1}{x}$$

(b) (6 points) $g(x) = \left(\sin^{-1}(5x)\right)^3$

$$g'(x) = 3\left(\sin^{-1}(5x)\right)^2 \cdot \frac{1}{\sqrt{1-(5x)^2}} \cdot 5$$

5. Consider the following differential equation

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

(a) (8 points) Find the general solution to this equation.

Already in standard form

$P = -\frac{1}{5x} \Rightarrow \int P\,dx = \int \frac{-1}{5x}dx = -\frac{1}{5}\ln|x|$

Integrating factor is $v = e^{\int P dx} = e^{-\frac{1}{5}\ln|x|} = e^{\ln(|x|^{-1/5})} = |x|^{-1/5}$

Choose $v = x^{-1/5}$.

$$x^{-1/5}y' - \frac{1}{5x}x^{-1/5}y = x\,x^{-1/5}$$

$$x^{-1/5}y' - \frac{1}{5}x^{-6/5}y = x^{4/5}$$

$$\frac{d}{dx}(x^{-1/5}y) = x^{4/5}$$

$$x^{-1/5}y = \int x^{4/5}dx = \frac{5}{9}x^{9/5} + C$$

$$y = \frac{5}{9}x^2 + Cx^{1/5}$$

(b) (2 points) Find the particular solution given the initial condition $y(1) = 1$.

$x = 1, y = 1$

$$1 = \frac{5}{9}(1)^2 + C \cdot 1^{1/5}$$

$$1 = \frac{5}{9} + C$$

$$\frac{4}{9} = C$$

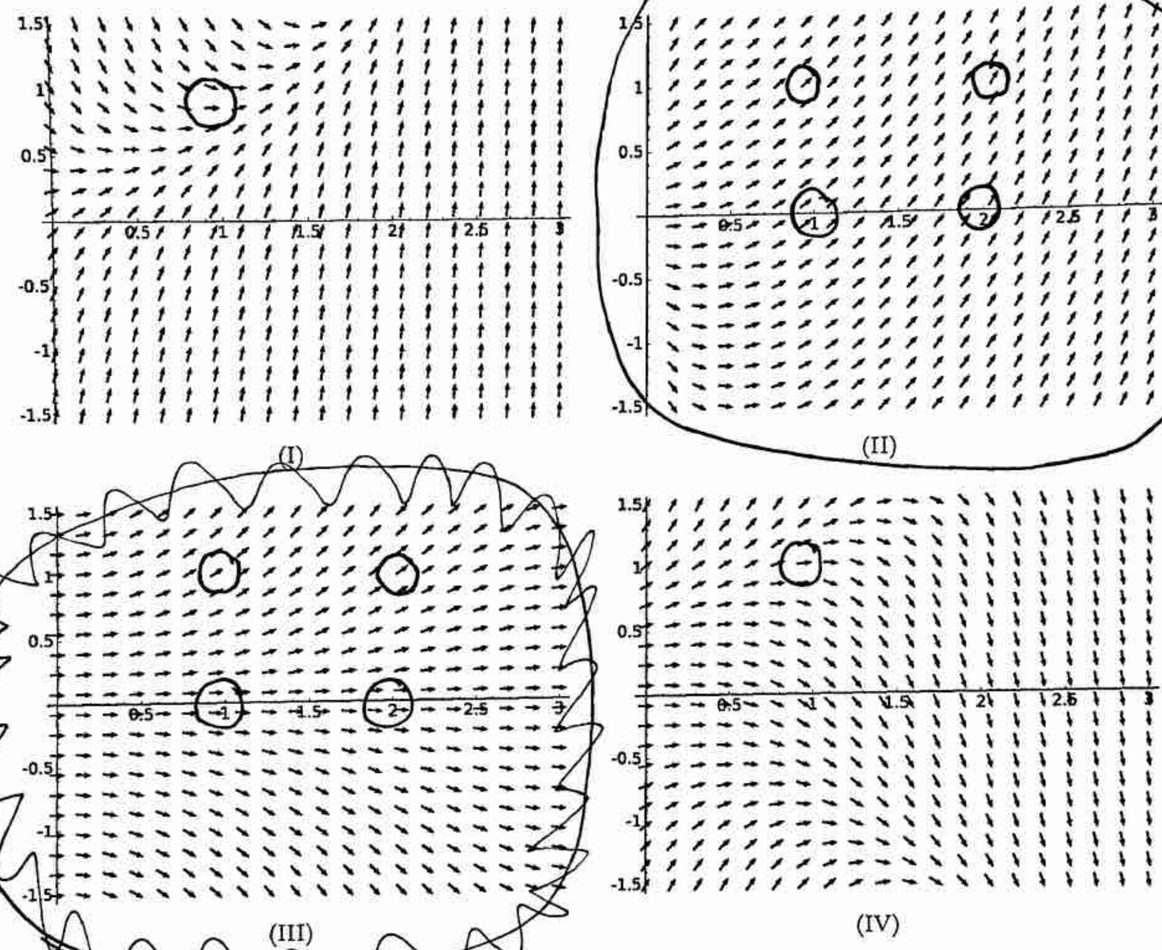
$$y = \frac{5}{9}x^2 + \frac{4}{9}x^{1/5}$$

(c) (3 points) Which of the following plots represents the slope field of this differential equation? That is, of the equation

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

Answer

Circle your answer.



(I) (II) (III) (IV)

WHOOPS

$y' = \frac{y}{5x} + x$		(I)	(II)	III	IV
(1,1)	$6/5$	≈ 0			≈ 0
(2,1)	$\frac{1}{10} + 2$				
(1,0)	1			0	
(2,0)	2			0	

not consistent

not consistent

so only II is possible.

6. In this problem, you will use numerical integration to estimate $\ln(2) = \int_1^2 \frac{dx}{x}$.

(a) (4 points) Graph the function $y = 1/x$ between $x = 1$ and $x = 2$. Draw on your graph the trapezoids used to apply the Trapezoidal Rule with $n = 2$. (So, your graph should have 2 trapezoids.)

$n = 2$

$\Delta x = \frac{2-1}{n} = \frac{2-1}{2} = \frac{1}{2}$

(b) (4 points) Use the Trapezoidal Rule with $n = 2$ to estimate $\ln 2$.

x_i:	1	$\frac{3}{2}$	2
$f(x_i)$	1	$\frac{2}{3}$	$\frac{1}{2}$
pattern	1	2	1

$\text{sum} = 1 \cdot 1 + 2 \cdot \frac{2}{3} + 1 \cdot \frac{1}{2} = 1 + \frac{4}{3} + \frac{1}{2} = \frac{6+8+3}{6}$

$= \frac{17}{6}$

$T_2 = \frac{\Delta x}{2}(\text{sum}) = \frac{\frac{1}{2}}{2}\left(\frac{17}{6}\right) = \frac{17}{24}$

(c) (2 points) Does the Trapezoidal Rule overestimate or underestimate $\ln 2$?

overestimate

7. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

(a) (7 points) Find its interval of convergence.

Absolute Root test

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{n\,3^n}\right|} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}\,3} = \frac{|x|}{3}$$

OR Absolute Ratio test

$$\rho = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)3^{n+1}} \Big/ \frac{x^n}{n\,3^n}\right| = \lim_{n\to\infty} \frac{|x|}{3}\frac{n}{n+1} = \frac{|x|}{3}.$$

In either method, solve $\rho<1$ to get $\frac{|x|}{3}<1 \iff |x|<3.$

$-3 < x < 3$ converges absolutely here

Check endpoints $x=3$: $\sum \frac{3^n}{n\,3^n} = \sum \frac{1}{n}$ diverges

$x=-3$: $\sum \frac{(-3)^n}{n\,3^n} = \sum \frac{(-1)^n}{n}$ converges by alternating series test because $\frac{1}{n}\to 0$ and is decreasing.

But $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ diverges, so $\sum \frac{(-1)^n}{n}$ converges conditionally

Interval of convergence

$[-3, 3)$

(b) (2 points) For what x does the series converge absolutely?

$-3 < x < 3$

8. (5 points) Evaluate the following limit.

Set $L = \lim_{t\to 0}(1+t)^{\frac{1}{2t}}$

$$\ln L = \lim_{t\to 0} \frac{1}{2t}\ln(1+t)$$

$$= \lim_{t\to 0} \frac{\ln(1+t)}{2t} \qquad \text{Type } \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{t\to 0} \frac{\frac{1}{1+t}}{2}$$

$$\ln L = \frac{1}{2}$$

Then $L = e^{1/2}$

9. (5 points) Consider the order 2 Taylor polynomial for $\ln(1+x)$ centered at $a=0$:

$$\ln(1+x) \approx x - \frac{x^2}{2}.$$

Use the Taylor remainder estimation theorem to estimate the error in this approximation when $|x| < 0.1$.

By formula sheet,

$$|R_2(x)| \leq \frac{M_3 |x-0|^3}{3!}$$

where $M_3 \geq$ max of $|f'''(x)|$ for $|x| < 0.1$

Here $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2} = -(1+x)^{-2}$$

$$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$$

For $-.1 < x < .1$, we have $|f'''(x)| \leq \frac{2}{(1-.1)^3} = \frac{2}{.9^3}$

We may use $M_3 = \frac{2}{.729}$

Then $|R_2(x)| \leq \frac{\left(\frac{2}{.729}\right)|x|^3}{3!} \leq \frac{\left(\frac{2}{.729}\right)(.1)^3}{6}$.

10. (6 points) What is the Taylor polynomial of order 3 for the function $f(x) = \sin(x)\cos(x)$ centered at $a = 0$?

$f(x) = \sin x \cos x \qquad f(0) = 0$

$f'(x) = \cos x \cos x + \sin x(-\sin x)$
$= \cos^2 x - \sin^2 x \qquad f'(0) = 1 - 0 = 1$

$f''(x) = 2\cos x(-\sin x) - 2\sin x \cos x$
$= -4\sin x \cos x \qquad f''(0) = 0$

$f'''(x) = -4\cos x \cos x - 4\sin x(-\sin x)$
$= -4\cos^2 x + 4\sin^2 x \qquad f'''(0) = -4$

$P_3(x) = 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 - \frac{4}{3!}(x-0)^3$
$= x - \frac{2}{3}x^3.$

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x \\
\sin^2 x &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\
\sin x \cos x &= \frac{1}{2}\sin(2x) \\
\sin x \sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y) \\
\cos x \cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y) \\
\sin x \cos y &= \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)
\end{aligned}$$

- Integrals of trigonometric functions.

$$\begin{aligned}
\int \tan x \; dx &= \ln|\sec x| + C \\
\int \cot x \; dx &= \ln|\sin x| + C \\
\int \sec x \; dx &= \ln|\sec x + \tan x| + C \\
\int \csc x \; dx &= -\ln|\csc x + \cot x| + C
\end{aligned}$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a, b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a, b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.