Math 242 Final Spring 2018

Name: Solutions

by Kenny.

(Please circle your section)

Section 1, MWF 9:30-10:20, Piper Harron

Section 2, MWF 9:30-10:20, Piper Harron

Section 3, MWF 11:30-12:20, Jeffrey Lyons

Section 4, MWF 11:30-12:20, Jeffrey Lyons

Section 5, TR 9:00-10:15, Luca Candelori

Section 6, TR 9:00-10:15, Luca Candelori

Section 7, TR 12:00-1:15, Luca Candelori

Section 8, TR 12:00-1:15, Luca Candelori

Section 9, TR 1:30-2:45, Yohsuke Watanabe

Section 10, TR 1:30-2:45, Yohsuke Watanabe

Page	Points	Score
2	20	
3	30	
4	10	
5	15	
6	15	
7	25	
8	25	
9	15	
10	45	
12	15	
13	50	
Total:	265	

- You may *not* use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it), except for the "short-answer" problems at the end.
- If you run out of room on a problem, continue on one of the blank pages after Page 13 and indicate clearly that you are doing this.
- On problems with choices (like Problem 12 do not do more parts than we ask you to, or you might be penalized!
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- **Don't forget to put your name on the paper!**
- Good luck!

1. (10 points) Find $f'(x)$ where $f(x) = \dfrac{2^x}{x^3\sqrt{1+x^2}}$.
(Hint: use logarithmic differentiation.)

$$\ln f(x) = \ln\left(\frac{2^x}{x^3\sqrt{1+x^2}}\right) = \ln(2^x) - \ln\left(x^3\sqrt{1+x^2}\right)$$

$$= \ln(2^x) - \ln(x^3) - \ln\sqrt{1+x^2}$$

$$= x\cdot\ln 2 - 3\ln x - \frac{1}{2}\ln(1+x^2)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln 2 - \frac{3}{x} - \frac{1}{2}\frac{1}{1+x^2}(2x)$$

$$\Rightarrow f'(x) = \frac{2^x}{x^3\sqrt{1+x^2}}\left(\ln 2 - \frac{3}{x} - \frac{x}{1+x^2}\right)$$

2. (10 points) $\lim\limits_{t\to 0} \dfrac{\cos(t)-\cos(3t)}{t^2} = ?$ Type $\frac{0}{0}$

$$\overset{H}{=} \lim_{t\to 0} \frac{-\sin(t) + 3\sin(3t)}{2t} \quad \text{Type } \frac{0}{0}$$

$$\overset{H}{=} \lim_{t\to 0} \frac{-\cos t + 9\cos(3t)}{2}$$

$$= \frac{-1+9}{2}$$

$$= 4$$

3. (30 points) Evaluate the integrals (on this page and the next).

(a) $\int \frac{\sin x dx}{2+\cos x}$

$$= \int \frac{-du}{u}$$

$u = 2+\cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= -\ln|u| + C$$

$$= -\ln|2+\cos x| + C$$

(b) $\int z\tan^{-1}(z)dz = ?$ Parts

$u = \tan^{-1} z \qquad dv = z\,dz$

$du = \frac{1}{z^2+1} \qquad v = \frac{z^2}{2}$

$$= \frac{z^2}{2}\tan^{-1} z - \frac{1}{2}\int \frac{z^2}{z^2+1}\,dz$$

$$= \frac{z^2}{2}\tan^{-1} z - \frac{1}{2}\int 1 - \frac{1}{z^2+1}\,dz$$

$$= \frac{z^2}{2}\tan^{-1} z - \frac{1}{2}\left[z - \tan^{-1} z\right] + C$$

(c) $\int \sin^3 x \cos^2 x dx = ?$ $= \int \sin^2 x \cos^2 x \sin x \, dx$

ODD power sin

$u = \cos x$

$du = -\sin x \, dx$

$\sin^2 x = 1 - \cos^2 x$

$= 1 - u^2$

$$= \int (1-u^2)\, u^2 \, (-du)$$

$$= \int -u^2 + u^4 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

4. (10 points) Estimate $\int_1^3 \frac{x}{1+x^2} dx$ using Simpson's rule with $n = 4$. **Don't simplify your answer!**

$a = 1, \; b = 3, \; n = 4, \; \Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 1/2$

x_i	1	3/2	2	5/2	3
$y_i = \frac{x}{1+x^2}$	1/2	6/13	2/5	20/29	3/10
S_4 coeffs	1	4	2	4	1
product	1/2	$4 \cdot 6/13$	$2 \cdot 2/5$	$4 \cdot \frac{20}{29}$	3/10

$$S_4 = \frac{1/2}{3}\left(1/2 + 4 \cdot \frac{6}{13} + 2 \cdot \frac{2}{5} + 4 \cdot \frac{20}{29} + \frac{3}{10}\right)$$

5. (15 points) Evaluate the integral: $\displaystyle\int \frac{-2x+3}{(x+1)(x^2-2x+2)}dx$

$b^2-4ac = 4-4\cdot 1\cdot 2 < 0 \implies x^2-2x+2$ is irreducible

Partial Fractions

$$\frac{-2x+3}{(x+1)(x^2-2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+2}$$

$$-2x+3 = A(x^2-2x+2) + (Bx+C)(x+1)$$

$$= Ax^2 - 2Ax + 2A + Bx^2 + Bx + Cx + C$$

$$= (A+B)x^2 + (-2A+B+C)x + (2A+C)$$

$$A+B = 0 \implies A = -B$$

$$-2A+B+C = -2 \implies \begin{cases} 2B+B+C = -2 \\ -2B \quad +C = 3 \end{cases} \text{ subtract}$$

$$2A+C = 3$$

$$5B = -5$$

$$B = -1 \implies A = 1$$

$$\implies 2+C = 3$$

$$C = 1$$

$$\int \frac{1}{x+1} + \frac{-x+1}{x^2-2x+2}\,dx$$

complete the square (not necessary, a regular u-sub works)

$$= \int \frac{1}{x+1} + \frac{-(x-1)}{(x-1)^2+1}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{u}{u^2+1}\,du$$

$$= \ln|x+1| - \frac{1}{2}\ln\left((x-1)^2+1\right) + C$$

6. (15 points) Evaluate the integral: $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

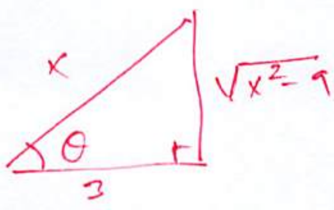
trig sub

$x = 3\sec\theta$

$x^3 = 27\sec^3\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-9} = 3\tan\theta$



$\sec\theta = \frac{x}{3} = \frac{\text{hyp}}{\text{adj}}$

$\cos\theta = \frac{3}{x}$

$\sin\theta = \frac{\sqrt{x^2-9}}{x}$

$$= \int \frac{\cancel{3}\tan\theta}{\underset{3}{\cancel{27}}\sec^{\cancel{3}\,2}\theta}\cdot\cancel{3}\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{3}\int \sin^2\theta\,d\theta$$

$$= \frac{1}{6}\int 1-\cos 2\theta\,d\theta$$

$$= \frac{1}{6}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{6}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{6}\left[\sec^{-1}\left(\frac{x}{3}\right) - \frac{3}{x}\cdot\frac{\sqrt{x^2-9}}{x}\right] + C$$

7. (15 points) Evaluate, or show diverges: $\int_1^\infty \frac{e^{-1/x}}{x^2} dx$

$$= \lim_{t \to \infty} \int_1^t \frac{e^{-1/x}}{x^2} dx$$

$u = -1/x$

$du = \frac{1}{x^2} dx$

$u(1) = -1$

$u(t) = -1/t$

$$= \lim_{t \to \infty} \int_{-1}^{-1/t} e^u \, du$$

$$= \lim_{t \to \infty} e^u \Big|_{-1}^{-1/t}$$

$$= \lim_{t \to \infty} \left[e^{-1/t} - e^{-1} \right] = 1 - \frac{1}{e}$$

8. (10 points) Set up a partial fractions decomposition for

$1^2 - 4 \cdot 1 \cdot 1 < 0$

careful

$$\frac{x^2 - 1}{(x^2 - 4)(2x + 1)^2(x^2 + x + 1)}$$

irred.

Do **not** find the relevant constants, just write down a sum of simple rational functions with unknown coefficients whose sum equals the rational function above.

$$\frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{2x+1} + \frac{D}{(2x+1)^2} + \frac{Ex+F}{x^2+x+1}$$

9. (15 points) Test $\sum_{n=2}^{\infty} \frac{(-1)^n}{-1+\sqrt{n}}$ for (a) absolute and (b) conditional convergence. (c) How many terms would we need to compute the sum with an error of at most $\frac{1}{99}$?

(1) $\sum_{n=2}^{\infty} \left| (-1)^n \frac{1}{-1+\sqrt{n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ comparison test

$\frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}}$ for $n \geq 2$

since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series $p = 1/2$), then $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges

(2) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt{n}-1}$ Alternating series test: $b_n = \frac{1}{\sqrt{n}-1}$

yes, b_n's are positive & decreasing, and

$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{-1/\sqrt{n}+1} = \frac{\sqrt{0}}{\sqrt{0}+1} = 0$

$\boxed{\therefore \text{ Series converges conditionally}}$

Alternating series estimation thm: $|R_{999}| < b_{1000} = \frac{1}{\sqrt{10000}-1} = \frac{1}{99}$

$\boxed{n = 999}$ terms

10. (10 points) Test ONLY ONE of the following two series for convergence. **Cross out the one you do not want graded:**

(a) $\sum_{n=0}^{\infty} \frac{7^n}{n!}$ (b) $\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{3n^2-1} \right)^n$

(a) Ratio test:

$\left| \frac{7^{n+1}}{(n+1)!} \cdot \frac{n!}{7^n} \right| = \frac{7}{n+1} \to 0 < 1$

$\therefore$ series converges

(b) Root test:

$\sqrt[n]{|a_n|} = \sqrt[n]{\left| \frac{2n^2+1}{3n^2-1} \right|^n}$

$= \frac{2n^2+1}{3n^2-1} \to \frac{2}{3} < 1$

$\therefore$ Series converges.

11. (15 points) Find the **radius** and **interval** of convergence for $\sum_{n=0}^{\infty} \frac{3n^2}{e^n} x^n$

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3(n+1)^2 x^{n+1}}{e^{n+1}} \cdot \frac{e^n}{3(n)^2 x^n}\right| = \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{e} \cdot |x|$$

$$\longrightarrow \frac{|x|}{e}$$

$$\frac{|x|}{e} < 1 \iff |x| < e \iff -e < x < e$$

$x = -e$: $\sum_{n=0}^{\infty} \frac{3n^2}{e^n} \cdot (-e)^n = \sum_{n=0}^{\infty} (-1)^n 3n^2$

diverges by the divergence test:

odd terms $\to -\infty$

even terms $\to +\infty$

$x = e$: $\sum_{n=0}^{\infty} \frac{3n^2}{e}(e)^n = \sum_{n=0}^{\infty} 3n^2$ diverges by divergence test, $3n^2 \to \infty$.

Radius $= e$

I.O.C. $= (-e, e)$

12. (45 points) Solve THREE of the following FOUR problems. DO NOT DO ALL FOUR!. Cross out the one you do **not** want graded.

(Continue your answers on the next page.)

~~(a) Solve the initial value problem: $y'' - 4y' + 4y = 0$; $y(0) = 1, y'(0) = 0$~~

(b) Find the general solution to: $\frac{dy}{dx} = -y\ln(x)$ Separable

(c) Find the general solution to: $t\frac{dy}{dt} + 2y = t^3$ linear

(d) In a certain culture of bacteria, the number of bacteria increased sixfold in 10 hours. How long did it take for their number to double?

(a) $\frac{1}{-y}dy = \ln x\, dx$ $\quad u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$-\int \frac{1}{y}\, dy = \int \ln x\, dx$

$-\ln|y| = x\ln x - \int dx = x\ln x - x + C$

$\ln|y| = x - x\ln x + C$

$|y| = e^{x - x\ln x + C} = e^{C} \cdot \frac{e^x}{x}$

$y = \pm e^{C} \cdot \frac{e^x}{x}$ (y = 0 is also a solution)

$\boxed{y = A \cdot \frac{e^x}{x}}$

(b) $y' + \frac{2}{t}y = t^2$

$p(t) = 2/t$

Integrating factor:

$e^{\int p(t)dt} = e^{\int 2/t\, dt} = e^{2\ln|t|} = t^2$

$(t^2 y)' = t^4$

$t^2 y = \int t^4 dt = \frac{t^5}{5} + c$

$\Rightarrow y = \frac{1}{t^2}\left(\frac{t^5}{5} + c\right)$

(c) $y(0) = y_0$

$y(10) = 6y_0 = y_0 e^{k \cdot 10}$

$\Rightarrow 6 = e^{k \cdot 10}$

$\ln 6 = k \cdot 10$

$k = (\ln 6)/10$

$2y_0 = y(t) = y_0 e^{\ln 6/10 \cdot t}$

$2 = e^{\ln 6/10\, t}$

$\frac{\ln 6}{10} t = \ln 2$

$\Rightarrow t = \frac{10\ln 2}{\ln 6}$

(Problem 12 continued)

$$\begin{array}{r|l} & \quad -1 \\ \hline 1+x & -x+1 \\ & -(-x-1) \\ \hline & \qquad 2 \end{array}$$

$$f(x) = \frac{1-x}{1+x} = -1 + \frac{2}{1+x}$$

$$= -1 + \frac{2}{1-(-x)}$$

$$= -1 + \sum_{n=0}^{\infty} 2(-1)^n x^n$$

$$= 1 + \sum_{n=1}^{\infty} 2(-1)^n x^n$$

13. (15 points) Find ONE of the following TWO series. DO NOT DO BOTH! Cross out the one you do **not** want graded.

   (a) Find the Taylor series for $f(x) = \ln(x)$ at $x = e$. (Note: the first 5 terms will get you most of the credit)

   (b) Find the MacLaurin series for $f(x) = \dfrac{1-x}{1+x}$. (Hint: what is the series for $\frac{1}{1+x}$?)

$f(x) = \ln x \implies f(e) = 1$

$f'(x) = \frac{1}{x} \implies f'(e) = \frac{1}{e} = (-1)^{1-1} \cdot \frac{1}{e^1}$

$f''(x) = -\frac{1}{x^2} \implies f''(e) = -\frac{1}{e^2} = (-1)^{2-1} \cdot \frac{1}{e^2}$

$f'''(x) = \frac{2}{x^3} \implies f'''(e) = \frac{2}{e^3} = (-1)^{3-1} \frac{2 \cdot 1}{e^3}$

$f^{(4)}(x) = \frac{-6}{x^4} \implies f^{(4)}(e) = \frac{-6}{e^4} = (-1)^{4-1} \frac{3 \cdot 2 \cdot 1}{e^4}$

$\vdots$

$f^{(n)}(e) = (-1)^{n-1} \frac{(n-1)!}{e^n}$

$$T(x) = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-e)^n}{n \cdot e^n}$$

previous page for (b)

The problems on this page are short answer. You need not show work. They should all be *very quick*, if not you are doing it wrong!

14. (20 points) Circle all series that converge.

A. $\sum_{n=1}^{\infty}(\frac{n-3}{n})$ B. $\sum_{n=3}^{\infty}\frac{\ln(n)}{n^3}$ C. $\sum_{n=3}^{\infty}\frac{5n^3-3n}{n^2(n-2)(n^2+5)}$ D. $\sum_{n=0}^{\infty}(\sqrt{2})^n$ E. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

15. (20 points) Short Answers. Put your answer in the space provided:

(a) $\log_2 64 =?$ $\log_2(2^6)$ (a) 6

(b) $\sum_{n=0}^{\infty}\frac{3^n}{2^{2n}} =$ $\sum_{n=0}^{\infty}(\frac{3}{4})^n$ (b) $\frac{1}{1-3/4}$

(c) $\sum_{k=1}^{\infty}(\frac{1}{k^2}-\frac{1}{k^2+2k+1}) =$ $\sum_{k=1}^{\infty}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)$ (c) 1

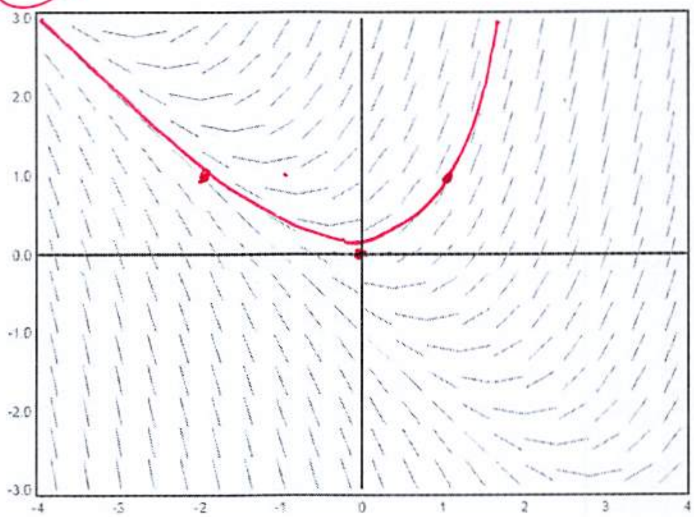
(d) $\lim_{n\to\infty}(1+\frac{3}{n})^n =$ (d) e^3

(e) $\lim_{n\to\infty}(\sqrt[n]{n}) =$ (e) 1

16. (10 points) (a) For the pictured slope field, circle the differential equation for which it is most likely the slope field:

A. $\frac{dy}{dx}=2xy$ B. $\frac{dy}{dx}=x+y$ C. $\frac{dy}{dx}=2y/x$ D. $\frac{dy}{dx}=2x/y$

	$(-2,1)$	$(0,0)$	$(1,1)$
A	-4	0	2
B	-1	0	2
C	-4	und.	2
D	-4	und	2



(b) On the same picture, sketch the solution to this equation that passes through the point $(1,1)$

(This page intentionally left blank. Use for scratch work or to finish a solution from earlier in the exam.)

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x \\
\sin^2 x &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\
\sin x \cos x &= \frac{1}{2}\sin(2x) \\
\sin x \sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y) \\
\cos x \cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y) \\
\sin x \cos y &= \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)
\end{aligned}$$

- Integrals of trigonometric functions.

$$\int \tan x\, dx = \ln|\sec x| + C$$
$$\int \cot x\, dx = \ln|\sin x| + C$$
$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\Big(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\Big)$$
$$S = \frac{\Delta x}{3}\Big(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\Big)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.