Math 242 Final

Name: ______________________ **Section:** ________

Instructor: ______________________ **Recitation Time:** ________

Solutions by Kenny.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	16	
7	17	
8	14	
9	14	
10	8	
11	7	
12	7	
Total:	133	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. (10 points) Evaluate $\int_0^{\pi/4} x\sin(5x)\,dx$

$$u = x \qquad dv = \sin 5x\,dx$$

$$du = dx \qquad v = -\frac{1}{5}\cos 5x$$

$$= -\frac{x}{5}\cos 5x\Big|_0^{\pi/4} - \int_0^{\pi/4}\left(-\frac{1}{5}\cos 5x\right)dx$$

$$= \left[-\frac{\pi}{20}\cos\frac{5\pi}{4} + 0\right] + \frac{1}{25}\sin 5x\Big|_0^{\pi/4}$$

$$= -\frac{\pi\sqrt{2}}{40} + \frac{1}{25}\left(\sin\frac{5\pi}{4} - 0\right)$$

$$= \frac{-\pi\sqrt{2}}{40} - \frac{\sqrt{2}}{50}$$

2. (10 points) Evaluate the following integral or show that it diverges.

$$\int_3^\infty \frac{1}{x(2x-1)}\,dx$$

$$= \lim_{t\to\infty} \int_3^t \frac{1}{x(2x-1)}\,dx \qquad \text{Partial fractions}$$

$$\frac{1}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1} \qquad \text{clear fractions}$$

$$1 = A(2x-1) + Bx \qquad \text{Clever \#'s : } 0,\ 1/2$$

$$x = 0: \quad 1 = A(2\cdot 0 - 1) + B\cdot 0 \implies A = -1$$

$$x = 1/2: \quad 1 = A(2\cdot 1/2 - 1) + B\cdot \tfrac{1}{2} \implies B = 2$$

$$= \lim_{t\to\infty} \int_3^t \frac{-1}{x} + \frac{2}{2x-1}\,dx$$

$$= \lim_{t\to\infty} \Big[-\ln|x| + \ln|2x-1|\Big]_3^t$$

$$= \lim_{t\to\infty} \ln\left|\frac{2x-1}{x}\right|\Bigg]_3^t$$

$$= \lim_{t\to\infty} \left(\ln\left|\frac{2t-1}{t}\right| - \ln 5/3\right)$$

$$= \ln\left|\lim_{t\to\infty} \frac{2t-1}{t}\right| - \ln 5/3$$

$$= \ln 2 - \ln 5/3$$

$$= \ln 6/5$$

3. (10 points) Evaluate the following integral. (Hint: Use trig substitution.)

$$\int \frac{1}{(1+x^2)^2}\,dx,$$

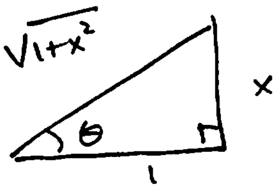
Pattern $1+x^2$ use tan.

$x = 1 \cdot \tan\theta$

$dx = \sec^2\theta \, d\theta$

$1+x^2 = 1+\tan^2\theta$

$= \sec^2\theta$



$\tan 6 = \frac{opp}{adj} = \frac{x}{1}$

$\sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{1+x^2}}$

$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sqrt{1+x^2}}$

$$= \int \frac{1}{(\sec^2\theta)^2} \sec^2\theta \, d\theta$$

$$= \int \frac{1}{\sec^2\theta} \, d\theta$$

$$= \int \cos^2\theta \, d\theta$$

EVEN power of cos use

$$= \frac{1}{2}\int 1+\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\tan^{-1}x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right] + C.$$

4. For each of the following series decide if it converges or diverges and explain why.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{n^4 - 2n^2}{n^5 + n}$ limit compare with $\sum_{n=1}^{\infty} \frac{n^4}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n \to \infty} \frac{n^4 - 2n^2}{n^5 + n} \cdot \frac{n}{1} = \lim_{n \to \infty} \frac{n^5 - 2n^3}{n^5 + n} \cdot \frac{1/n^5}{1/n^5}$$

$$= \lim_{n \to \infty} \frac{1 - 2/n^2}{1 + 1/n^4} = \frac{1 - 0}{1 + 0} = 1 = c$$

Since $0 < c < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series $p = 1$), then the series $\sum_{n=1}^{\infty} \frac{n^4 - 2n^2}{n^5 + n}$ diverges

(b) (5 points) $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ use integral test: $f(x) = \frac{1}{x(\ln x)^2}$

f is pos, dec, cts, interpolates, and

$$\int_3^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \to \infty} \int_3^t \frac{1}{x(\ln x)^2} dx$$

$u = \ln x$

$du = \frac{1}{x} dx$

$u(t) = \ln t$

$u(3) = \ln 3$

$$= \lim_{t \to \infty} \int_{\ln 3}^{\ln t} \frac{1}{u^2} du$$

$$= \lim_{t \to \infty} \left. \frac{u^{-1}}{-1} \right|_{\ln 3}^{\ln t}$$

$$= \lim_{t \to \infty} \left(\frac{1}{\ln 3} - \frac{1}{\ln t} \right) = \frac{1}{\ln 3}$$

$\therefore$ series converges

5. For each of the following series, determine its sum.

(a) (5 points) $\sum_{n=0}^{\infty}(-1)^n \frac{1}{3^n}$ geometric

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$$

$$\frac{\text{first term}}{1-\text{ratio}}$$

$$= \frac{(-1/3)^0}{1-(-1/3)}$$

$$= \frac{1}{1+1/3}$$

$$= \frac{3}{4}$$

(b) (5 points) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ Taylor series

$$= e^{-1}$$

6. Find the derivative of each of the following functions.

(a) (8 points) $f(x) = (\ln x)^x$

logarithmic differentiation

$$\ln f(x) = \ln\left[(\ln x)^x\right] = x \cdot \ln \ln x$$

$$\Longrightarrow \frac{f'(x)}{f(x)} = 1 \cdot \ln \ln x + x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$\Longrightarrow f'(x) = (\ln x)^x \left[\ln \ln x + \frac{1}{\ln x} \right]$$

(b) (8 points) $g(x) = \left(\sin^{-1}(7x)\right)^4$

Chain rule

$$g'(x) = 4 \cdot \left(\sin^{-1}(7x)\right)^3 \cdot \frac{1}{\sqrt{1-(7x)^2}} \cdot 7$$

7. Consider the following differential equation

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

(a) (8 points) Find the general solution to this equation.

$P(x) = -\frac{1}{5x}$

Integrating factor : $I = e^{\int P(x)\,dx} = e^{\int -\frac{1}{5x}\,dx} = e^{-\frac{1}{5}\ln|x|} = |x|^{-1/5}$

$= x^{-1/5}$

$x^{-1/5}y' - x^{-1/5}\left(-\frac{1}{5x}\right)y = x^{-1/5}\cdot x$

$\left(x^{-1/5}\cdot y\right)' = x^{4/5}$

$x^{-1/5}y = \int x^{4/5}\,dx$

$y = x^{1/5}\left(\frac{x^{9/5}}{9/5} + c\right) = x^{1/5}\left(5/9\, x^{9/5} + c\right)$

(b) (3 points) Find the particular solution given the initial condition $y(1) = 0$.

$0 = y(1) = 1^{1/5}\left(5/9 \cdot 1^{9/5} + c\right)$

$= 5/9 + c$

$\Rightarrow c = -5/9$

$\Rightarrow y = x^{1/5}\left(5/9\, x^{9/5} + (-5/9)\right)$

(c) (3 points) Which of the following plots represents the direction field of this differential equation? That is, of the equation

$(1,1) \quad y' = \frac{1}{5} + 1 = \frac{6}{5}$

$\Rightarrow$ I & IV wrt

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

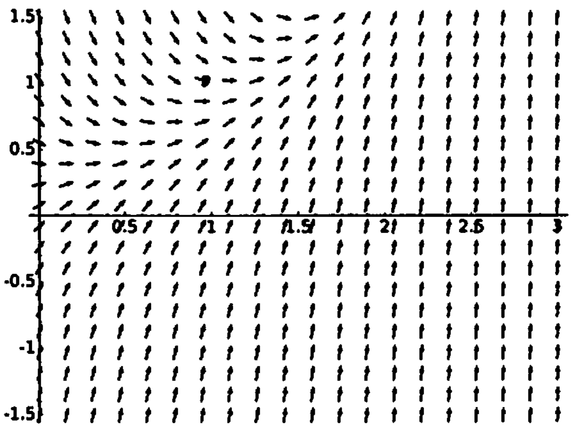
Circle your answer.

$y' = \frac{1}{5x} y + x$

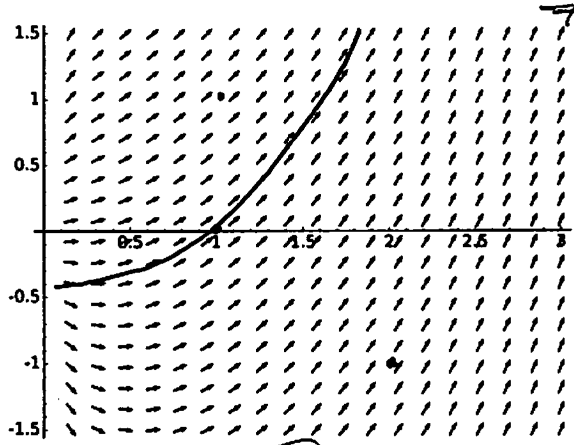
$(2,-1) \quad y' = -\frac{1}{10} + (-2) =$ ~~[illegible]~~

$\frac{19}{10}$

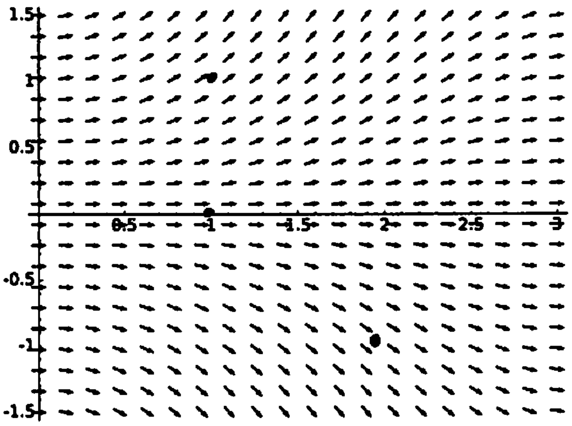
$\Rightarrow$ III wrt



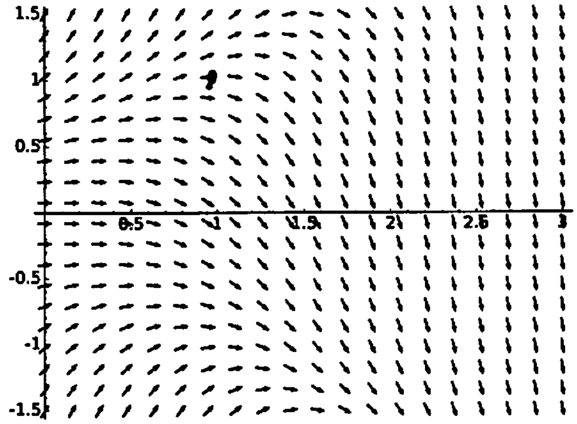
(I)



(II) answer



(III) incorrect

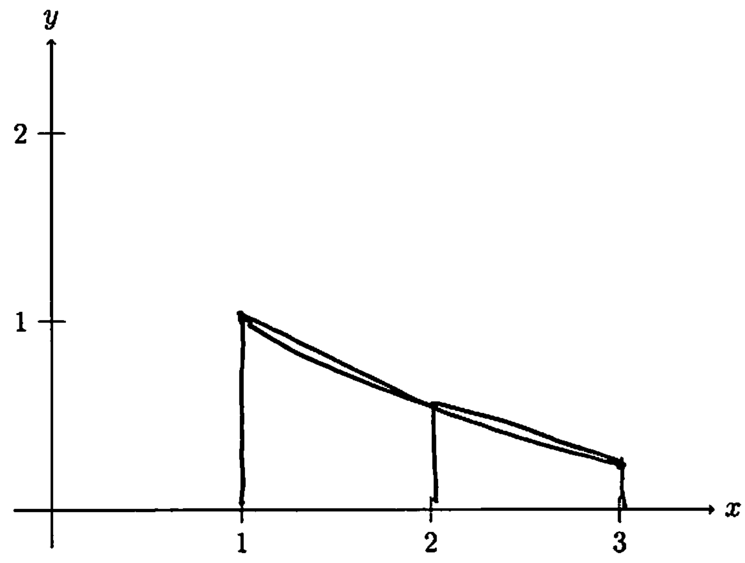


(IV)

(d) (3 points) Sketch the the particular solution to this differential equation that satisfies $y(1) = 0$ on the direction field you choose in part (c).

8. In this problem, you will use numerical integration to estimate $\ln(3) = \int_1^3 \frac{dx}{x}$. (The formula sheet has both the formula for numerical integration and the error estimates.)

(a) (4 points) Graph the function $y = 1/x$ between $x = 1$ and $x = 3$. Draw on your graph the trapezoids used to apply the Trapezoidal Rule with $n = 2$. (So, your graph should have 2 trapezoids.)



(b) (2 points) Does the Trapezoidal Rule overestimate or underestimate the value of ln(3)?

overestimate

(c) (4 points) Use the Trapezoidal Rule with $n = 2$ to estimate ln(3).

$a = 1,\ b = 3,\ n = 2,\ \Delta x = \frac{b-a}{n} = 1$

x:	1	2	3
y:	1	1/2	1/3
T_2 coeff	1	2	1
product	1	1	1/3

$\}$ mult.

Sum $= 1 + 1 + 1/3$

$T_2 = \frac{\Delta x}{2} \cdot \text{Sum}$

$= \frac{1}{2} \cdot (1 + 1 + 1/3)$

(d) (4 points) Use the error estimate to give an upper bound on the absolute value of the error.

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

f'' is positive and decreasing on $[1,3]$ so is largest at $x = 1$.

$|f''(x)| \leq |f''(1)| = |2 \cdot 1^{-3}| = 2$

we may use $M = 2$.

$|E_T| \leq \frac{M(b-a)^3}{12 \cdot n^2} = \frac{2 \cdot 2^3}{12 \cdot 2^2}$

9. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$.

(a) (10 points) Find its interval of convergence.

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} 4^{n+1} x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n 4^n \cdot x^n}\right| = 4|x| \cdot \sqrt{\frac{n}{n+1}}$$

$$\to 4|x|\sqrt{\frac{1}{1+0}} = 4|x|$$

Series converges (absolutely) for

$4|x| < 1 \iff |x| < 1/4 \iff x \in (-1/4, 1/4)$

Endpt $x = -1/4$: $\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{\sqrt{n}} \left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

(p-series $p = 1/2$).

Endpt $x = 1/4$: $\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{\sqrt{n}} \left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$ doesn't converge in absolute value (see above), but does converge by AST ($b_n = \frac{1}{\sqrt{n}}$ is pos, dec, $b_n \to 0$).

(b) (4 points) For which values of x does the series converge absolutely?

I.O.C $= (-1/4, 1/4]$

the series converges absolutely for $(-1/4, 1/4)$.

10. (8 points) Evaluate the following limit.

$$L = \lim_{t \to 0}(1+t)^{\frac{1}{3t}} \qquad \text{Type } 1^{\infty}$$

$$\ln L = \lim_{t \to 0} \frac{\ln(1+t)}{3t} \qquad \text{Type } \frac{0}{0}$$

$$\overset{H}{=} \lim_{t \to 0} \frac{\frac{1}{1+t}}{3}$$

$$= \lim_{t \to 0} \frac{1}{3(1+t)} \qquad \text{sub rule.}$$

$$= \frac{1}{3(1+0)}$$

$$= \frac{1}{3}$$

$$\therefore \; L = e^{1/3}$$

11. (7 points) Consider the degree 2 Taylor polynomial for $\ln(1+x)$ centered at $a = 0$:

$$\ln(1+x) \approx x - \frac{x^2}{2}.$$

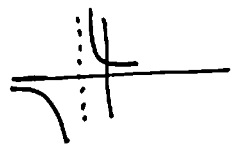
Use the Taylor remainder estimation theorem to estimate the error in this approximation when $|x| < 0.1$.

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f''(x) = -(1+x)^{-2}$$

$$f'''(x) = 2(1+x)^{-3}$$



$$f^{(4)}(x) = -6(1+x)^{-4}$$

f''' is positive and decreasing ($f^{(4)} < 0$) for $|x| < 0.1$

So is largest at left endpt. $x = -0.1$

$$|f'''(x)| \le |f'''(-0.1)| = 2(1+0.1)^{-3}$$
$$= 2(1.1)^{-3}$$

we may use $M = 2(1.1)^{-3}$.

$$|R_2(x)| \le M \cdot \frac{|x|^3}{3!} \le \frac{2(1.1)^{-3} \cdot (0.1)^3}{3!}$$

12. (7 points) What is the Taylor polynomial of degree 3 for the function $f(x) = \dfrac{1}{1+x^3}$ centered at $a = 0$?

Hard way:

$$f(x) = \frac{1}{1+x^3} = (1+x^3)^{-1} \implies f(0) = 1$$

$$f'(x) = -(1+x^3)^{-2} \cdot 3x^2 \implies f'(0) = 0$$

$$f''(x) = 2(1+x^3)^{-3}(3x^2)^2 + (-1)(1+x^3)^{-2} \cdot 6x \implies f''(0) = 0$$

$$f'''(x) = -6(1+x^3)^{-4}(3x^2)^3 + 2(1+x^3)^{-3} \cdot 2(3x^2)' \cdot 6x$$
$$- (-2)(1+x^3)^{-3} \cdot 3x^2 \cdot 6x - (1+x^3)^{-2} \cdot 6$$

$$\implies f'''(0) = -6$$

$$T_3(x) = 1 + 0 + 0 - \frac{6}{3!}x^3 = 1 - x^3$$

Easy way.

$$f(x) = \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n$$
$$= 1 - x^3 + x^6 - \cdots$$

$$\therefore \; T_3(x) = 1 - x^3.$$

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$
$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$
$$\sin x \cos x = \frac{1}{2}\sin(2x)$$
$$\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$
$$\cos x \cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$
$$\sin x \cos y = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

- Integrals of trigonometric functions.

$$\int \tan x \, dx = \ln|\sec x| + C$$
$$\int \cot x \, dx = \ln|\sin x| + C$$
$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.