Exam 1 Review

September 24, 2022

Inverse Functions:

- Know the process of finding inverses.
- Know how to graph the inverse given the graph of the original function.
- Know the differentiation rule
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Exponential and Logarithmic Functions

- Definition: $\log_a(y) = x \;\Leftrightarrow\; a^x = y$.
- Limits:
$$\lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to -\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$$
- Derivatives:
$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln a} \qquad \frac{d}{dx}a^x = a^x \ln a$$
- Integrals:
$$\int e^x \; dx = e^x + C \qquad \int a^x \; dx = \frac{a^x}{\ln a} + C$$
$$\int \frac{1}{x} \; dx = \ln|x| + C$$
- Know the procedure for logarithmic differentiation, and when it is useful.
- Exponential growth/decay:
  - The model applies when the rate of change of a quantity is proportional to the current amount. In symbols, $y' = ky$.
  - The general solution is always: $y = y_0 e^{kt}$.
  - Always write down your givens, and don't forget units in your answers.
  - Most problems require you to determine y_0 and k. The strategy for finding them depends on the information given in the problem.

Inverse Trig. Functions

- Definitions:

$$\cos^{-1}: [-1.1] \to [0, \pi] \qquad \cos^{-1}(y) = \theta \;\Leftrightarrow\; \cos(\theta) = y$$
$$\sin^{-1}: [-1.1] \to [-\pi/2, \pi/2] \qquad \sin^{-1}(y) = \theta \;\Leftrightarrow\; \sin(\theta) = y$$
$$\tan^{-1}: (-\infty, \infty) \to (-\pi/2, \pi/2) \qquad \tan^{-1}(y) = \theta \;\Leftrightarrow\; \tan(\theta) = y$$

- Limits:
$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2} \qquad \lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$$

- Derivatives:

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

Limits of Indeterminate Forms

- Type $\frac{0}{0}$ or Type $\frac{\infty}{\infty}$: use L'Hôpital's rule.
- Type $0 \cdot \infty$: use $ab = \frac{b}{1/a}$.
- Type $\infty - \infty$: Try to combine into one expression using algebra, identities, etc.
- Type 0^0, 1^∞, and ∞^0: Set the limit equal to L, take ln, then $\ln L$ is of one of the types above. Careful, the answer is e^L, not L.

Integration by Parts

- Integration by parts formula: $\int udv = uv - \int vdu$.
- Generally speaking, your choice of u should be something simple to differentiate, and dv should be something simple to integrate.
- The integral $\int vdu$ needs to be "easier" to solve than $\int udv$.
- The mnemonic LIATE (Logarithms, Inverse Trig, Algebraic, Trigonometric, Exponential) is helpful for choosing u.

Trigonometric Integrals

- Sine and Cosine (same angle): $\int \sin^m x \cos^n x dx$
  - Odd power of sine: let $u = \cos x$, then $du = -\sin x dx$ and $\sin^2 x = 1 - \cos^x = 1 - u^2$.
  - Odd power of cosine: let $u = \sin x$, then $du = \cos x dx$ and $\cos^2 x = 1 - \sin^x = 1 - u^2$.
  - Both powers even: use half-angle identities

$$\cos^2 x = \frac{1+\cos 2x}{2}, \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

- Tangent and Secant: $\int \tan^m x \sec^n x dx$
  - Odd power of tangent: let $u = \sec x$, then $du = \sec x \tan x dx$ and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$

- Even power of secant: let $u = \tan x$, then $du = \sec^2 x dx$ and $\sec^2 x = 1 + \tan^2 x = 1 + u^2$.
- Even power of tangent and odd power of secant: these are tougher and very very vey rarely are ever tested. Convert everything to secant using $\tan^2 + 1 = \sec^2 x$ and use the secant reduction formula (we will not cover this).

- Products of cotangent and cosecant are handled similarly as tangent and secant.
- Sine and Cosine (different angle): $\int \sin(mx)\cos(nx)dx$. Use the identities:

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

- For other types of expressions, rewrite in terms of sine and cosine.

Trigonometric Substitution

- Pattern $a^2 - x^2$: use the substitution $x = a \sin\theta$.
- Pattern $x^2 - a^2$: use the substitution $x = a \sec\theta$.
- Pattern $a^2 + x^2$: use the substitution $x = a \tan\theta$.
- After perfoming the substitution you should be left with a trigonometric integral (apply the above techniques).
- To return the result in terms of θ back to x use the right triangle that is determined by the substitution

$$\sin\theta = \frac{x}{a} = \frac{opp}{hyp}, \qquad \sec\theta = \frac{x}{a} = \frac{hyp}{adj}, \qquad \tan\theta = \frac{x}{a} = \frac{opp}{adj}$$

Partial Fraction Decomposition

- Technique for integrating rational functions $\displaystyle\int \frac{p(x)}{q(x)}\,dx$ where p and q are polynomials.
- If $\deg p \geq \deg q$, do long division first. Then do partial fraction decomposition on the remainder.
- To do partial fraction decomposition:
  1. Factor the denominator completely into a product of irreducible factors. The degree of each factor must be 1 or 2. A quadratic is irreducible if the *discriminant* $b^2 - 4ac$ is negative.
  2. Write out the partial fraction decomposition abstractly.
     - Linear factors get one unknown constant A.
     - Quadratic factors get two unknown constants $Ax + B$.
     - If there are repeated factors, then the denominators have increasing powers.
  3. Solve for the unknown constants:
     (a) Clear fractions.
     (b) Combine like terms.
     (c) Equate coefficients.
     (d) Solve this system of equations.
  4. Finally, integrate each fraction individually.