Name: Section: 5 6 9 10

1. Solve the initial value problem:

$$y'\sqrt{9-x^2} = (y^2+1)x^2$$

$$\frac{dy}{y^2+1} = \frac{x^2}{\sqrt{9-x^2}}\,dx$$

$$\int \frac{dy}{y^2+1} = \int \frac{x^2}{\sqrt{9-x^2}}\,dx$$

$x = 3\sin x$

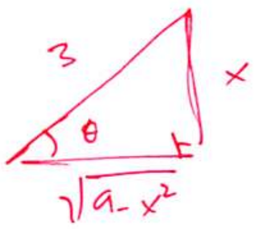
$x^2 = 9\sin^2 x$

$dx = 3\cos x\,dx$

$\sqrt{9-x^2} = 3\cos x$

$$\tan^{-1} y = \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta$$

$$= \frac{9}{2}\int 1-\cos 2\theta\,d\theta$$



$$= \frac{9}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} - \sin\theta\cos\theta\right] + C$$

$$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right] + C$$

$$\Rightarrow \quad y = \tan\left(\frac{9}{2}\left[\sin^{-1}\frac{x}{3} - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right] + C\right)$$

2. Newton's law of cooling states

$$\frac{dT}{dt} = k(T - T_s),$$

where T is the temperaure of an object, T_s is the temperature of its surroundings, and k is a constant.

(a) Solve this ODE.

(b) A cup of tea starts at $210°F$ in a room at $71°F$. After 10 minutes the tea cools to $140°F$. What is the temperature of the cup of tea after another 10 minutes?

(a) $\frac{1}{T - T_s}\, dT = k\,dt$

$\int \frac{1}{T-T_s}\, dT = \int k\,dt$

$\ln|T - T_s| = kt + c$

$|T - T_s| = e^{kt+c}$

$T - T_s = \pm e^{c} \cdot e^{kt}$

Since $T = T_s$ is a solution we can write

$T = T_s + A \cdot e^{kt}, \quad A \in \mathbb{R}$

(b) $T(0) = 210, \quad T_s = 71$

$\Rightarrow 210 = 71 + A e^{k \cdot 0}$

$210 = 71 + A$

$A = 139$

$\therefore T = 71 + 139 e^{k \cdot t}$

Since $T(10) = 140$, then

$140 = 71 + 139 e^{k \cdot 10}$

$69 = 139 e^{k \cdot 10}$

$\frac{69}{139} = e^{k \cdot 10}$

$k \cdot 10 = \ln {}^{69}/_{129}$

$k = \frac{1}{10} \ln {}^{69}/_{139}$

$\therefore$

$T(20) = 71 + 139 e^{\frac{1}{10}\ln {}^{69}/_{139} \cdot 20}$