Math 242 Homework 1: due 6/17

1. Recall that a function is 1-1 if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Sketch the following functions and determine if they are 1-1 on their domain. If the function is not 1-1, find two points x_1 and x_2 such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

   (a) $f(x) = 1/x$

   (b) $f(x) = x^4 + x^2$

   (c) $f(x) = \begin{cases} -|x| & x < 0 \\ 1 & x = 0 \\ 1/x^2 & x > 0 \end{cases}$

2. Let $f(x) = x^2 - 2x + 1$ for $x \geq 1$.

   (a) Compute f^{-1}. [*hint*: use the quadratic formula].

   (b) Sketch f and f^{-1} on the same graph.

   (c) State the domain and range of f^{-1}.

3. Let $f(x) = 1/x$.

   (a) Show that f is its own inverse.

   (b) Sketch the graph of f. What do you think we can say about a 1-1 functions whose graph is symmetric about the diagonal $y = x$?

   (c) Use IFT to compute the derivative of f^{-1}.

4. Let $f(x) = 4x^2$ for $x \geq 0$.

   (a) Determine the inverse of f.

   (b) Sketch f and f^{-1} on the same graph.

   (c) Show that

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=5}}.$$

5. Let $f(x) = (2x - 5)^{-1}(x^2 - 5x)^6$ for $x > 5$. Determine the value of the derivative of f^{-1} at the point $x = f(6)$. [*hint*: you do not need to calculate a formula for f^{-1}].

6. Suppose g is an invertible, differentiable function whose graph passes through the origin with slope 3. Find the slope of the graph of g^{-1} as it passes through the origin.

7. Let f be an increasing function. It's not too difficult to see that f is 1-1. It's clear by vertical line test, but let's use the definition of 1-1 to show it. Suppose $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_2 < x_1$. Since f is increasing the first case implies $f(x_1) < f(x_2)$, in other words, $f(x_1) \neq f(x_2)$. Similar reasoning holds in the other case. Thus, any increasing function is invertible! Maybe not that exciting, but we will use this fact in this problem.

   (a) Show that $f(x) = (x - 1)^3$ is increasing on its domain.

   (b) Compute f^{-1}.

(c) Compute f' and find the points x such that $f'(x) = 0$.

(d) The above discussion together with part (a) say that f is invertible. If we exclude all the points you find in part (c), then we can apply the IFT to find a formula for $(f^{-1})'$. Do it.

(e) Confirm your calculation from the previous part by computing the derivative of f^{-1} directly.

8. Compute the derivative of the following functions.

(a) $y = t(\ln t)^2$

(b) $y = \ln x^3$

(c) $y = (\ln x)^3$

(d) $y = t\sqrt{\ln t}$

(e) $y = (\ln \theta)^\pi$

(f) $y = \log_5 e^x$

(g) $y = \ln(3\theta e^{-\theta})$

(h) $y = e^\theta(\cos\theta + \sin\theta)$

(i) $y = \cos(e^{-x^2})$

(j) $y = x^\pi$

9. Evaluate the following integrals.

(a) $\displaystyle\int_2^4 \frac{dx}{x(\ln x)^2}$

(b) $\displaystyle\int \frac{8r}{4r^2 - 5} dr$

(c) $\displaystyle\int \frac{\sec y \tan y}{2 + \sec y} dy$

(d) $\displaystyle\int_0^{\pi/12} 6\tan(3x) dx$

(e) $\displaystyle\int \frac{dx}{2\sqrt{x} + 2x}$

(f) $\displaystyle\int \frac{dx}{1 + e^x}$

(g) $\displaystyle\int \frac{e^{-1/x^2}}{x^3} dx$

(h) $\displaystyle\int_1^2 \frac{2^{\ln x}}{x} dx$

(i) $\displaystyle\int_1^e x^{\ln 2 - 1} dx$

(j) $\displaystyle\int_1^{e^x} \frac{1}{t} dt$

10. Find y' using logarithmic differentiation or implicit differentiation.

(a) $y = \tan\theta\sqrt{2\theta + 1}$

(b) $y = \theta\sin\theta(\sec\theta)^{-1/2}$

(c) $y = (\ln x)^{\ln x}$

(d) $\ln y = e^y \sin x$

(e) $\ln xy = e^{x+y}$

(f) $\tan y = e^x + \ln x$

11. Solve the initial value problem $y' = e^t \sin(e^t - 2)$ with initial condition $y(\ln 2) = 0$.

12. Find the area between the curve $y = 2^{1-x}$ and the x-axis between the points -1 and 1. Sketch the graph.

13. Define the functions f and g by

$$f(x) = \frac{e^x - e^{-x}}{2} \text{ and } g(x) = \frac{e^x + e^{-x}}{2}.$$

(a) Show that $f(x)^2 - g(x)^2 = 1$.

(b) Show that $f'(x) = g(x)$.

(c) Show that $\displaystyle\int f(x)dx = g(x) + C$.