(1.e) $y = \cot^{-1}\frac{1}{x} - x\tan^{-1}x$

$$y' = \frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right) - \frac{d}{dx}\left(x\tan^{-1}x\right)$$

$$= -\frac{d}{dx}\tan^{-1}\frac{1}{x} - \left[\tan^{-1}x + x\,\frac{1}{1+x^2}\right]$$

$$= -\frac{1}{\left(\frac{1}{x}\right)^2+1}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) - \tan^{-1}x - \frac{x}{1+x^2}$$

$$= \frac{1}{\frac{1}{x^2}+1}\cdot\frac{1}{x^2} - \tan^{-1}x - \frac{x}{1+x^2}$$

$$= \frac{1}{1+x^2} - \tan^{-1}x - \frac{x}{1+x^2}$$

$$= \frac{1-x}{1+x^2} - \tan^{-1}x$$

(2.c) $I = \int e^{-x} \cos 2x \, dx$, apply parts.

$$u = e^{-x} \qquad dv = \cos 2x \, dx$$
$$du = -e^{-x} dx \qquad v = \tfrac{1}{2} \sin 2x$$

$$\Rightarrow \quad I = \tfrac{1}{2} e^{-x} \sin 2x - \int \left(\tfrac{1}{2} \sin 2x\right)\left(-e^{-x}\right) dx$$

$$= \tfrac{1}{2} e^{-x} \sin 2x + \tfrac{1}{2} \int e^{-x} \sin 2x \, dx$$

Apply parts again.

$$u = e^{-x} \qquad dv = \sin 2x \, dx$$
$$du = -e^{-x} dx \qquad v = -\tfrac{1}{2} \cos 2x$$

$$\Rightarrow \quad I = \tfrac{1}{2} e^{-x} \sin 2x + \tfrac{1}{2}\left[-\tfrac{1}{2} e^{-x} \cos 2x - \int \left(-\frac{\cos 2x}{2}\right)\left(-e^{-x} dx\right)\right]$$

$$= \tfrac{1}{2} e^{-x} \sin 2x - \tfrac{1}{4} e^{-x} \cos 2x - \tfrac{1}{4} \int e^{-x} \cos 2x \, dx$$

$$= \tfrac{1}{2} e^{-x} \sin 2x - \tfrac{1}{4} e^{-x} \cos 2x - \tfrac{1}{4} I$$

$$\Rightarrow \quad \tfrac{5}{4} I = \tfrac{1}{2} e^{-x} \sin 2x - \tfrac{1}{4} e^{-x} \cos 2x$$

$$\therefore \quad I = \tfrac{1}{5}\left(2 e^{-x} \sin 2x - e^{-x} \cos 2x\right)$$

$$= \frac{e^{-x}}{5}\left(2 \sin 2x - \cos 2x\right)$$

(2.c) using tabular integration

$$I = \int e^{-x} \cos 2x \, dx$$

u		dv	
e^{-x}	$+$	$\cos 2x$	
$-e^{-x}$	$-$	$\frac{1}{2}\sin 2x$	
e^{-x}		$-\frac{1}{4}\cos 2x$	stop

$$I = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x - \int \left(-\frac{1}{4}\cos 2x\right) e^{-x}\, dx$$

$$= \frac{e^{-x}}{4}\left(2\sin 2x - \cos 2x\right) + \frac{1}{4}\int e^{-x}\cos 2x \, dx$$

$$= \frac{e^{-x}}{4}\left(2\sin 2x - \cos 2x\right) + \frac{1}{4} I$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{-x}}{4}\left(2\sin 2x - \cos 2x\right)$$

$$I = \frac{e^{-x}}{5}\left(2\sin 2x - \cos 2x\right)$$

(2.i) $I = \int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

complete the square under the radical

$$x^2-4x+3 = x^2-4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+3$$
$$= \left(x^2-4x+4\right)-4+3$$
$$= (x-2)^2-1$$

$$\Longrightarrow \quad I = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}} \qquad \begin{array}{l} u = x-2 \\ du = dx \end{array}$$

$$= \int \frac{du}{u\sqrt{u^2-1}}$$

$$= \sec^{-1}|u| + C$$

$$= \sec^{-1}|x-2| + C$$

(2.l) $\quad I = \displaystyle\int_0^{1/2} \cos^{-1} x \, dx$ $\qquad$ parts

$u = \cos^{-1} x \qquad dv = dx$

$du = -\dfrac{1}{\sqrt{1-x^2}}\, dx \qquad v = x$

$$I = x\cos^{-1}x \Big|_0^{1/2} - \int_0^{1/2} (x)\left(-\frac{1}{\sqrt{1-x^2}}\, dx\right)$$

$$= x\cos^{-1}x \Big|_0^{1/2} + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} \qquad \begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \end{aligned}$$

$$= x\cos^{-1}x \Big|_0^{1/2} - \frac{1}{2}\int_1^{3/4} \frac{du}{\sqrt{u}}$$

$$= x\cos^{-1}x \Big|_0^{1/2} - \sqrt{u}\Big|_1^{3/4}$$

$$= \frac{1}{2}\cos^{-1}\frac{1}{2} - 0\cdot\cancel{\cos^{-1} 0} - \left(\sqrt{3/4} - \sqrt{1}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{3} + 1 - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$$

(3.b) $\lim\limits_{x\to 0^+} \sin x \cdot \ln x$ $\qquad 0\cdot\infty$

$$= \lim_{x\to 0^+} \frac{\ln x}{1/\sin x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

$$\stackrel{H}{=} \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x}$$

$$= \lim_{x\to 0^+} \frac{\sin^2 x}{x\cdot\cos x} \qquad \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x\to 0^+} \frac{2\sin x\cos x}{\cos x - x\sin x}$$

$$= \frac{2\cdot 0\cdot 1}{1 - 0\cdot 0}$$

$$= 0$$

(3.c) Let $f(x) = (e^x + x)^{1/x}$, then

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \ln (e^x + x)^{1/x}$$

$$= \lim_{x\to\infty} \frac{\ln(e^x+x)}{x} \qquad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}$$

$$= \lim_{x\to\infty} \frac{e^x+1}{e^x+x} \qquad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{e^x}{e^x+1} \qquad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 = L$$

$$\therefore \lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} f(x) = e^L = e^1 = e.$$

(4.a) $y = Ce^{x^3/3}$; $y' = x^2 y$

$$y' = Ce^{x^3/3} \cdot \frac{d}{dx}\left(\frac{x^3}{3}\right)$$
$$= Ce^{x^3/3} \cdot x^2$$
$$= x^2 \cdot y$$

(5.b) $\frac{dy}{dx} = \frac{xe^x}{y\sqrt{1+y^2}}$

$$\Longrightarrow y\sqrt{1+y^2}\,dy = xe^x\,dx$$

u-sub ↳ $\int y\sqrt{1+y^2}\,dy = \int xe^x\,dx$ ← parts

x	$+$	e^x
1	$-$	e^x
0		e^x

$u = 1+y^2$

$dy = 2y\,dy$

→

$$\frac{1}{2}\int \sqrt{u}\,du = xe^x - e^x + C$$

$$\frac{1}{3}u^{3/2} = xe^x - e^x + C$$

$$\frac{1}{3}(1+y^2)^{3/2} = xe^x - e^x + C$$

$$1+y^2 = \left(3e^x(x-1) + C\right)^{2/3}$$

$$y^2 = \left(3e^x(x-1) + C\right)^{2/3} - 1$$

$$y = \pm\sqrt{\left(3e^x(x-1) + c\right)^{2/3} - 1}$$

Leaving your solution in implicit form is OK.

6. $y = \sin(kt)$

$y' = k\cos(kt)$

$y'' = -k^2 \sin(kt)$

$\Longrightarrow \quad 9y'' = -4y$

$\Longleftrightarrow \quad -9k^2 \sin(kt) = 4\sin(kt)$

$\Longleftrightarrow \quad 0 = -4\sin(kt) + 9k^2 \sin(kt)$

$0 = (9k^2 - 4)\sin(kt)$

$0 = (3k-2)(3k+2)\sin kt$

$\Longleftrightarrow \quad 3k-2=0$ or $3k+2=0$ or $\sin kt = 0$

$k = \pm \frac{2}{3}$ or $kt = n\pi$ where n is any integer

$k = \pm \frac{2}{3}$ or $k = \frac{n\pi}{t}$, $n \in \mathbb{Z}$ and $t \neq 0$.

(8) $L(x) :=$ intensity of light

$x :=$ feet beneath the surface

$L_0 :=$ surface value

Since L satisfies $\frac{dL}{dx} = kL$, then

$$L(x) = L_0 e^{k \cdot x}$$

We are given that $L(18) = \frac{1}{2} L_0$

so that

$$\frac{1}{2} L_0 = L_0 e^{k \cdot 18}$$

$$\frac{1}{2} = e^{k \cdot 18}$$

$$k = \frac{\ln \frac{1}{2}}{18}$$

Want x such that $L(x) = \frac{1}{10} L_0$

$$\frac{1}{10} L_0 = L_0 e^{\frac{\ln \frac{1}{2}}{18} \cdot x} \implies \frac{1}{10} = e^{\frac{\ln \frac{1}{2}}{18} \cdot x}$$

$$\implies \ln \tfrac{1}{10} = \frac{\ln \frac{1}{2}}{18} \cdot x$$

$$\therefore x = \frac{18 \ln \frac{1}{10}}{\ln \frac{1}{2}}$$

(a) $\dfrac{dC}{dt} = r - kC$

Let $y = r - kC$, then

$$\frac{dy}{dt} = \frac{d}{dt}(r - kC)$$

$$= -k\frac{dC}{dt}$$

$$= -k(r - kC)$$

$$= -ky$$

$\therefore$ A solution to this differential equation $\left(\frac{dy}{dt} = -ky\right)$ is

$$y = y_0 e^{-k \cdot t} \quad (*)$$

where $y_0 = y(0) = r - kC_0$

Substitute $y = r - kC$ and $y_0 = r - kC_0$ into $(*)$ and solve for C.

$\longrightarrow$

$$r - kC = (r - kC_0)e^{-k \cdot t}$$

$$-kC = (r - kC_0)e^{-k \cdot t} - r$$

$$kC = r - (r - kC_0)e^{-kt}$$

$$C = \frac{1}{k}\left(r - (r - kC_0)e^{-kt}\right)$$

(10)

(a)
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\Rightarrow \quad \frac{dP}{P\left(1 - \frac{P}{K}\right)} = r\,dt$$

$$\Rightarrow \quad \int \frac{dP}{P(1 - P/K)} = \int r\,dt$$

$$\int \frac{1}{P}\,dP + \int \frac{1}{K-P}\,dP = \int r\,dt \qquad \text{(Hint)}$$

$$\ln|P| \cancel{+} - \ln|K-P| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C$$

$$\left|\frac{P}{K-P}\right| = e^{C} \cdot e^{rt}$$

$$\frac{P}{K-P} = \pm e^{c} \cdot e^{rt}$$

$\pm e^{a}$ is just a constant so let's just call it C.

$$\frac{P}{K-P} = C \cdot e^{rt}$$

$$P = C \cdot e^{rt}(K-P)$$

$$P = KCe^{rt} - Ce^{rt} \cdot P$$

$$+Ce^{rt} \cdot P \qquad\qquad +Ce^{rt} \cdot P$$

$$P + Ce^{rt} \cdot P = KCe^{rt}$$

$$P(1 + Ce^{rt}) = KCe^{rt}$$

$$P = \frac{KCe^{rt}}{1 + Ce^{rt}}$$

Now multiply the RHS by $1 = \frac{K}{K}$

$$P = \frac{K}{K} \cdot \frac{KCe^{rt}}{(1 + Ce^{rt})}$$

$$= \frac{K(KC)e^{rt}}{K + (KC)e^{rt}}$$

Again KC is just a constant so let's just call it C.

$$P = \frac{KCe^{rt}}{K + Ce^{rt}}$$

(b) Let $P(0) = P_0$, then find C

$$P_0 = \frac{KCe^{r \cdot 0}}{K + Ce^{r \cdot 0}}$$

$\longrightarrow$

$$P_0 = \frac{KC}{K+C} \qquad e^0 = 1.$$

$$\Longrightarrow P_0(K+C) = KC$$

$$P_0K + P_0C = KC$$

$$P_0K = KC - P_0C$$

$$= C(K - P_0)$$

$$\therefore C = \frac{P_0K}{K - P_0}$$

Hence,

$$P(t) = \frac{K\left(\frac{P_0K}{K-P_0}\right)e^{rt}}{K + \left(\frac{P_0K}{K-P_0}\right)e^{rt}}$$

To simplify multiply by $\frac{K-P_0}{K-P_0}$

Numerator:

$$K \frac{P_0 K}{K - P_0} e^{rt} \cdot (K - P_0)$$

$$= K (P_0 K) e^{rt}$$

Denominator:

$$\left[K + \frac{P_0 K}{K - P_0} e^{rt} \right] \cdot (K - P_0)$$

$$= K(K - P_0) + P_0 K e^{rt}$$

$$= K \left[K - P_0 + P_0 e^{rt} \right]$$

$$= K \left[K + P_0 (e^{rt} - 1) \right]$$

$$\therefore\ P(t) = \frac{\cancel{K}(P_0 K)e^{rt}}{\cancel{K}\left[K + P_0(e^{rt}-1)\right]}$$

$$= \frac{P_0 K e^{rt}}{K + P_0(e^{rt}-1)}$$

$$(2)\ \lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 K e^{rt}}{K + P_0(e^{rt}-1)} \qquad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{t\to\infty} \frac{P_0 K e^{rt}\cdot r}{P_0 e^{rt}\cdot r}$$

$$= \lim_{t\to\infty} K$$

$$= K$$