Math 242 Homework 2: due 6/24

1. Find the derivative of y with respect to the appropriate variable.

   (a) $y = \cos^{-1}(x^3)$

   (b) $y = \sin^{-1}(1 - t^2)$

   (c) $y = \sec^{-1}(x^2 + 5)$

   (d) $y = \csc^{-1}(e^t + 1)$,

   (e) $y = \cot^{-1}\frac{1}{x} - x\tan^{-1} x$

   (f) $y = \ln\tan^{-1} x$

2. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{dx}{\sqrt{1 - 9x^2}}$

   (b) $\displaystyle\int_{-2}^{2} \frac{dt}{4 + 3t^2}$

   (c) $\displaystyle\int e^{-x}\cos(2x)dx$

   (d) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x dx}{1 + e^{2x}}$

   (e) $\displaystyle\int x\sin(3x)dx$

   (f) $\displaystyle\int \frac{dx}{\sqrt{6x - x^2}}$

   (g) $\displaystyle\int \frac{3dy}{y^2 + 4y + 8}$

   (h) $\displaystyle\int \frac{(\sin^{-1} x)^3 dx}{\sqrt{1 - x^2}}$

   (i) $\displaystyle\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

   (j) $\displaystyle\int t^5 e^t dt$

   (k) $\displaystyle\int \frac{dx}{x\sqrt{3x^2 - 16}}$

   (l) $\displaystyle\int_0^{1/2} \cos^{-1} x dx$

3. Compute the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

   (b) $\displaystyle\lim_{x\to 0^+} \sin x \ln x$

   (c) $\displaystyle\lim_{x\to\infty} (e^x + x)^{1/x}$

   (d) $\displaystyle\lim_{\theta\to 0}(\csc\theta - \cot\theta)$

   (e) $\displaystyle\lim_{x\to 0^+} x^{x^2}$

   (f) $\displaystyle\lim_{t\to 0^+} \frac{\ln t}{t}$

4. In the following problems show that $y = f(x)$ is a solution to the given differential equation. In these problems C is a constant.

   (a) $y = \dfrac{3 + e^x}{3 - e^x}; \quad 2y' = y^2 - 1$

   (b) $y = (C - x^2)^{-1/2}; \quad y' = xy^3$

   (c) $y = Ce^{x^3/3}; \quad y' = x^2 y$

5. Solve the following differential equations.

   (a) $xy' = y$

   (b) $\dfrac{dy}{dx} = \dfrac{xe^x}{y\sqrt{1 + y^2}}$

   (c) $y' = (4 - y^2)^{1/2}$

   (d) $2\sin^{-1}(y)y' = x$

6. For which values k is $y = \sin(kt)$ a solution to the differential equation $9y'' = -4y$?

7. The processing of raw sugar has a step called "inversion" which changes the sugar's molecular structure. Once inversion begins, the rate of change of the amount of sugar is proportional the the amount of raw sugar remaining. If 2000 kg of raw sugar reduces to 1400 kg during the first 10 hours, how much raw sugar will remain after another 14 hours?

8. The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation
$$\frac{dL}{dx} = kL.$$
Eighteen feet below the surface cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. How deep can you expect to work without artificial light?

9. A glucose solution is administered to a patients bloodstream at a constant rate r. As the solution is added, the patient's body converts it into other substances at a rate proportional to the concentration at that time. A model for the concentration C of solution in the patient's bloodstream is
$$\frac{dC}{dt} = r - kC,$$
where k is a positive constant. Derive a formula for the concentration if at time $t = 0$ the concentration is C_0. [*Hint*: use the substitution $y = r - kC$ to solve the differential equation. We did this in class in the bison problem. If you were not there, see Newton's law of cooling in the text.]

10. The Verhulst model for population growth captures the idea that the rate of growth of a population is faster the further away it is from carrying capacity, and that the rate slows down the closer it gets to carrying capacity. In symbols, a population P satisfies the differential equation
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$
where K is the carrying capacity and r is the rate of growth. In this problem we will assume both K and r are constant.

    (a) Solve the above differential equation. Your answer should have the form
$$P(t) = \frac{KCe^{rt}}{1 + Ce^{rt}} \quad \text{or} \quad P(t) = \frac{KCe^{rt}}{K + Ce^{rt}}.$$
    These two functions are the same, however the constant C is different in each. [*Hint*: you will probably need to use the fact that
$$\frac{1}{P(1 - \frac{P}{K})} = \frac{1}{P} + \frac{1}{K - P}.$$
    We will learn how to break fractions down like this in section 8.4.]

    (b) If the population at time $t = 0$ is P_0, show that the formula which models the population's size is
$$P(t) = \frac{KP_0e^{rt}}{K + P_0(e^{rt} - 1)}.$$

(c) Show that as time goes on the population tends towards carrying capacity. That is,

$$\lim_{t \to \infty} P(t) = K.$$