(1.d) $\displaystyle\int_{-\infty}^{0} z\,2^z\,dz$ improper + parts

Evaluate $\displaystyle\int_{a}^{0} z\,2^z\,dz$

$$
\begin{array}{ccc}
z & & 2^z \\
 & + & \\
1 & & 2^z/\ln 2 \\
 & - & \\
0 & & 2^z/(\ln 2)^2
\end{array}
$$

$$\Longrightarrow \int_{a}^{0} z\,2^z\,dz = \left[\frac{z\,2^z}{\ln 2} - \frac{2^z}{(\ln 2)^2}\right]_{a}^{0}$$

$$= \left[\frac{0\cdot 2^0}{\ln 2} - \frac{2^0}{(\ln 2)^2}\right] - \left[\frac{a2^a}{\ln 2} - \frac{2^a}{(\ln 2)^2}\right]$$

$$= -\frac{1}{(\ln 2)^2} - \frac{a2^a}{\ln 2} - \frac{2^a}{(\ln 2)^2}$$

$$\therefore \int_{-\infty}^{0} z\,2^z\,dz = \lim_{a\to-\infty}\int_{a}^{0} z\,2^z\,dz$$

$$= \lim_{a\to-\infty}\left[-\frac{1}{(\ln 2)^2} - \frac{a2^a}{\ln 2} - \frac{2^a}{(\ln 2)^2}\right]$$

$$= -\frac{1}{(\ln 2)^2} \underbrace{- \lim_{a\to-\infty} \frac{a2^a}{\ln 2}}_{\text{this one needs some work.}} - \cancel{\lim_{a\to-\infty} \frac{2^a}{(\ln 2)^2}}^{0}$$

$$\lim_{a\to-\infty} \frac{a2^a}{\ln 2} = \frac{1}{\ln 2} \lim_{a\to-\infty} a2^a \qquad \infty\cdot 0$$

$$= \frac{1}{\ln 2} \lim_{a\to-\infty} \frac{a}{1/2^a} \qquad \frac{\infty}{\infty}$$

$$\overset{H}{=} \frac{1}{\ln 2} \lim_{a\to-\infty} \frac{1}{2^{-a}\cdot \ln 2 \cdot (-1)}$$

$$= -\frac{1}{(\ln 2)^2} \lim_{a\to-\infty} 2^a$$

$$= 0$$

$$\boxed{\therefore \int_{-\infty}^{0} z2^z\,dz = -\frac{1}{(\ln 2)^2}}$$

(1.g) $\displaystyle\int \frac{2x-1}{\sqrt{x^2+4x+5}}$

Complete the square:

$$x^2+4x+5 = (x+4x+4)+1$$
$$= (x+2)^2+1$$

$$\int \frac{2x-1}{\sqrt{(x+2)^2+1}}\,dx = \int \frac{2u-5}{\sqrt{u^2+1}}\,du$$

$u = x+2$

$\Rightarrow \cancel{u+2=x}$

$u-2 = x$

$2(u-2)-1 = 2x-1$

$2u-5 = 2x-1$

$du = dx$

$$= \int \frac{2u\,du}{\sqrt{u^2+1}} - \int \frac{5\,du}{\sqrt{u^2+1}} \quad (*)$$

$$= 2\int \frac{\tan\theta \sec^2\theta}{\sec\theta}\,d\theta - 5\int \frac{\sec^2\theta\,d\theta}{\sec\theta}$$

(*) trig sub

$u = \tan\theta$

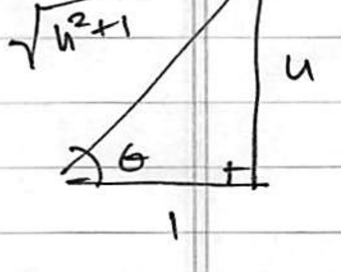
$u^2+1 = \sec^2\theta$

$du = \sec^2\theta\,d\theta$

$$= 2\int \tan\theta\sec\theta\,d\theta - 5\int \sec\theta\,d\theta$$

$$= 2\sec\theta - 5\ln|\sec\theta+\tan\theta| + C$$

$$= 2\sqrt{u^2+1} - 5\ln\left|\sqrt{u^2+1} + u\right| + C$$



$$\boxed{= 2\sqrt{(x+2)^2+1} - 5\ln\left|\sqrt{(x+2)^2+1} + (x+2)\right| + C}$$

(1.j) $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$ improper $s \neq 2$

$$= \int_0^2 \frac{s}{\sqrt{4-s^2}}\,ds + \int_0^2 \frac{1}{\sqrt{4-s^2}}\,ds$$

$$\int \frac{s}{\sqrt{4-s^2}}\,ds = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$u = 4-s^2$

$du = -2s\,ds \qquad = -\sqrt{u} + C$

$$\therefore \int_0^2 \frac{s}{\sqrt{4-s^2}}\,ds = \lim_{b \to 2^-} \int_0^b \frac{s}{\sqrt{4-s^2}}\,ds$$

$$= \lim_{b \to 2^-} -\sqrt{u}\,\Big|_0^b$$

$$= \lim_{b \to 2^-} -\sqrt{b} \qquad = 2$$

$$\int_0^2 \frac{1}{\sqrt{4-s^2}}\,ds = \lim_{b \to 2^-} \int_0^b \frac{1}{\sqrt{4-s^2}}\,ds$$

$$= \lim_{b \to 2^-} \sin^{-1}\left(\frac{s}{2}\right)\Big|_0^b$$

$$= \lim_{b \to 2^-} \sin^{-1}\left(\frac{b}{2}\right) - \sin^{-1}(0)$$

$$= \pi/2 - 0$$

$$= \pi/2$$

$$\therefore \int_0^2 \frac{s+1}{\sqrt{4-s^2}}\, ds = 2 + \pi/2$$

$$\boxed{= \frac{4+\pi}{2}}$$

(1.e) $\int_0^{1/3} \sqrt{1-9t^2}\, dt$ trig sub.

~~sub~~

$u = 3t$

$du = 3dt$

$\hookrightarrow = \frac{1}{3}\int_0^{1} \sqrt{1-u^2}\, du$

$= \frac{1}{3}\int_0^{\pi/2} \cos\theta \cos\theta\, d\theta$

$u = \sin\theta$

$1-u^2 = \cos^2\theta$

$0 = \sin\theta \iff \theta = 0$

$1 = \sin\theta \iff \theta = \pi/2$

$du = \cos\theta\, d\theta$

$= \frac{1}{3}\int_0^{\pi/2} \cos^2\theta\, d\theta$

$= \frac{1}{6}\int_0^{\pi/2} 1 + \cos 2\theta\, d\theta$

$= \frac{1}{6}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= \frac{1}{6}\left[\pi/2 + \frac{1}{2}\cdot 0 - 0\right)$

$= \pi/12$

(1.n) $\displaystyle\int_{-\pi/2}^{0} \frac{ds}{\cos s - 1}$ improper $s \neq 0$.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$2\sin^2\theta = 1-\cos 2\theta$$

$$\frac{1}{2\sin^2\theta} = \frac{1}{1-\cos 2\theta}$$

$$-2\csc^2\theta = \frac{1}{\cos 2\theta - 1}$$

Let $2\theta = s$, then

$$-2\csc^2\left(\frac{s}{2}\right) = \frac{1}{\cos s - 1}$$

$$\int_{-\pi/2}^{0} \frac{ds}{\cos s - 1} = -2\int_{-\pi/2}^{0} \csc^2\left(\frac{s}{2}\right) ds$$

$$= -2\lim_{b\to 0^-} \int_{-\pi/2}^{b} \csc^2\left(\frac{s}{2}\right) ds$$

$$= -2\lim_{b\to 0^-} -2\cot\left(\frac{s}{2}\right)\Big|_{-\pi/2}^{b}$$

$$= 4 \lim_{b \to 0^-} \cot\left(\frac{b}{2}\right) - \cot\left(\frac{-\pi}{4}\right)$$

$$= 4 \lim_{b \to 0^-} \overset{\nearrow \infty}{\frac{\cos\left(\frac{b}{2}\right)}{\sin\left(\frac{b}{2}\right)}} + 1$$

$$= \infty$$

(1.a) $\displaystyle\int_0^1 \frac{10\,dx}{(x-1)^2(x+2)}$

$$\frac{10}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$10 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$
$$= A(x^2+x-2) + B(x+2) + C(x^2-2x+1)$$

$$\begin{cases} A + C = 0 \\ A + B - 2C = 0 \\ -2A + 2B + C = 10 \end{cases} \rightarrow \begin{cases} A + C = 0 \\ -2A - 2B + 4C = 0 \\ -2A + 2B + C = 10 \end{cases}$$

$$\rightarrow \begin{cases} A + C = 0 \\ -2A - 2B + 4C = 0 \\ -4A + 5C = 10 \end{cases} \rightarrow \begin{cases} 4A + 4C = 0 \\ -2A - 2B + 4C = 0 \\ -4A + 5C = 10 \end{cases}$$

$$\rightarrow \begin{cases} 9C = 10 \\ -2A - 2B + 4C = 0 \\ -4A + 5C = 10 \end{cases} \Rightarrow C = 10/9$$

$\therefore A = -10/9$

$\therefore -10/9 + B - 2(10/9) = 0$

$B - 10/3 = 0$

$B = 10/3$

$$\int_0^1 \frac{10\,dx}{(x-1)^2(x+2)} = \frac{-10}{9}\int_0^1 \frac{dx}{x-1} + \frac{10}{3}\int_0^1 \frac{dx}{(x-1)^2} + \frac{10}{9}\int_0^1 \frac{dx}{x+2}$$

$$\frac{-10}{9}\int_0^1 \frac{dx}{x-1} = \frac{-10}{9} \lim_{b\to 1^-}\int_0^b \frac{dx}{x-1}$$

$$= \cancel{\frac{-10}{9}\Big[\ln|x-1|\Big]_0^b} \quad \frac{-10}{9}\lim_{b\to 1^-}\Big[\ln|x-1|\Big]_0^b$$

$$= \frac{-10}{9}\lim_{b\to 1^-} \overset{\to\infty}{\ln|b-1|} + \frac{10}{9}\ln|0-1|$$

$$\therefore \int_0^1 \frac{10\,dx}{(x-1)^2(x+2)} \text{ diverges}$$

(1.5) $\displaystyle\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$.

$$\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2}$$

$$\begin{aligned} y^2+2y+1 &= (Ay+B)(y^2+1) + Cy + D \\ &= Ay^3 + By^2 + Ay + B + Cy + D \end{aligned}$$

$$\begin{cases} A = 0 \\ B = 1 \\ A+C = 2 \\ D = 1 \end{cases} \longrightarrow \quad \begin{aligned} 0+C &= 2 \\ C &= 2 \end{aligned}$$

$A=0,\ B=1,\ C=2,\ D=1.$

$$\begin{aligned} \int \frac{y^2+2y+1}{(y^2+1)^2}\,dy &= \int \frac{dy}{y^2+1} + \int \frac{2y+1}{y^2+1}\,dy \\ &= 2\int \frac{dy}{y^2+1} + \int \frac{2y}{y^2+1}\,dy \\ &= 2\tan^{-1} y + \int \frac{du}{u} \\ &= 2\tan^{-1} y + \ln|y^2+1| + C. \end{aligned}$$

(2.d) $\displaystyle\int_0^\infty \frac{ds}{1+e^s}$ converges.

Note that $\displaystyle\int_0^\infty \frac{ds}{e^s} = \lim_{b\to\infty} \int_0^b \frac{ds}{e^s}$

$$= \lim_{b\to\infty} -\frac{1}{e^s}\Big]_0^b$$

$$= \lim_{b\to\infty} \overset{0}{\cancel{-\frac{1}{e^b}}} + \frac{1}{e^0}$$

$$= 1.$$

$$\lim_{s\to\infty} \frac{\frac{1}{1+e^s}}{\frac{1}{e^s}} = \lim_{s\to\infty} \frac{e^s}{1+e^s} \quad \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{s\to\infty} \frac{e^s}{e^s}$$

$$= 1$$

Since this limit exists and is nonzero then by the limit comparison test $\displaystyle\int_0^\infty \frac{ds}{1+e^s}$ converges as $\displaystyle\int_0^\infty \frac{ds}{e^s}$ does

(3.a) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

pf. Let $a \in \mathbb{R}$. and let f be odd.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \begin{aligned} u &= -x \\ du &= -dx \end{aligned}$$

$$= -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \qquad f \text{ is odd}$$

$$= 0.$$

Infinite case is the same, just put a limit infront of every term.

$\cancel{\int_{-\infty}^{\infty} f(x)\,dx} =$

(3.b)

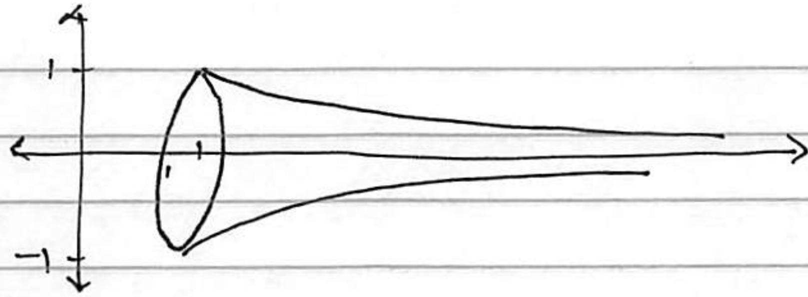
f	g	$f \circ g$
even	even	even
even	odd	even
odd	even	even
odd	odd	odd

(3.c) $\displaystyle\int_{-a}^{a} \sin^k x \, dx = 0$ whenever $k \geq 1$ is odd

$f(x) = x^k$ and $g(x) = \sin x$ are both odd. Then part (b) says $(f \circ g)(x) = \sin^k x$ is odd. Therefore, by part (a)

$$\int_{-a}^{a} \sin^k x \, dx = 0.$$

(4.a)



typical area element $A(x) = \pi \cdot \text{radius}^2$

$= \pi \cdot \left(\frac{1}{x}\right)^2$

$$\text{Volume} = \int_1^\infty \pi \cdot \frac{1}{x^2}\,dx = \pi \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{converges } (p=2)}$$

$\therefore$ Volume is finite.

(4.b)

typical circumference element

$C(x) = 2\pi \cdot \text{radius}$

$= 2\pi \cdot \frac{1}{x}$

$$\text{Surface Area} = \int_1^\infty \frac{2\pi}{x}\,dx$$

$$= 2\pi \underbrace{\int_1^\infty \frac{1}{x}\,dx}_{\text{diverges } (p=1)}$$

$\therefore$ Surface area is infinite.

(5a) Let $a > 0$ and let n be integer that follows a. ($n = \lceil a \rceil$).
then

$$\lim_{x\to\infty} \frac{x^a}{e^x} \qquad \frac{\infty}{\infty}$$

$$\stackrel{H}{=} \lim_{x\to\infty} \frac{a x^{a-1}}{e^x} \qquad \frac{\infty}{\infty}$$

n iterations of L'Hôpital's rule $\vdots$

$$\stackrel{H}{=} \lim_{x\to\infty} \frac{a(a-1)(a-2)\cdots(a-(n-1))\, x^{a-n}}{e^x}$$

$a-n<0$ since $a<n$

$$= \lim_{x\to\infty} \frac{a(a-1)(a-2)\cdots(a-(n-1))}{x^{n-a} e^x}$$

$= 0$

(6.a) $\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\,dx$

$= \int_0^\infty \frac{1}{e^x}\,dx = 1$ (see solution to problem 2.d)

(6.b) $\Gamma(n+1) = \int_0^\infty x^{(n+1)-1} e^{-x}\,dx$

$= \int_0^\infty x^n e^{-x}\,dx$

$u = x^n \qquad dv = e^{-x}\,dx$

$du = nx^{n-1}\,dx \qquad v = -e^{-x}$

$\rightarrow \Gamma(n+1) = uv\Big|_0^\infty - \int_0^\infty v\,du$

$= -x^n e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})\,nx^{n-1}\,dx$

$= \underbrace{\lim_{b\to\infty} -x^n e^{-x}\Big|_0^b}_{0} + \underbrace{n\int_0^\infty x^{n-1} e^{-x}\,dx}_{\Gamma(n)}$

$= n\,\Gamma(n).$

(6.c) Let n be a positive integer
Then using part (b) over and over

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$= n(n-1)\,\Gamma(n-1)$$

$$= n(n-1)(n-2)\,\Gamma(n-2)$$

$$= \vdots$$

$$= n(n-1)(n-2)\cdots 2\,\Gamma(2)$$

$$= n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1)$$

$$= n(n-1)(n-2)\cdots 2\cdot 1\cdot 1 \quad \text{part (a)}$$

$$= n!$$

(6.d)

$$\Gamma(1/2) = \int_0^\infty x^{1/2 - 1} e^{-x}\, dx$$

$$= \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx \qquad u = \sqrt{x} \Rightarrow u^2 = x$$

$$du = \frac{1}{2\sqrt{x}}\, dx$$

$$= 2\int_0^\infty e^{-u^2}\, du$$

Note that e^{-u^2} is even so that

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2}\, du = \int_{-\infty}^\infty e^{-u^2}\, du$$

$$= \sqrt{\pi}$$