Math 242 Homework 3: due 7/1

1. Evaluate the following integrals

   (a) $\int_1^2 x^{-2} \ln x dx$

   (b) $\int \frac{x^3 + x^2}{x^2 + 5x + 6} dx$

   (c) $\int_{-\pi}^{\pi} \sin^5 x dx$

   (d) $\int_{-\infty}^{0} z 2^z dz$

   (e) $\int_{-\infty}^{\infty} 2x e^{-x^2} dx$

   (f) $\int_0^{\pi} \sin^4 \theta \cos^2 \theta d\theta$

   (g) $\int \frac{2x - 1}{\sqrt{x^2 + 4x + 5}} dx$

   (h) $\int \sin \ln x dx$

   (i) $\int \frac{2x^2 + x - 8}{x^3 + 4x} dx$

   (j) $\int_0^2 \frac{s + 1}{\sqrt{4 - s^2}} ds$

   (k) $\int_0^{\pi/4} x\sqrt{1 - \cos 2x} dx$

   (l) $\int_0^{1/3} \sqrt{1 - 9t^2} dt$

   (m) $\int \tan^3 x \sqrt{\sec x} dx$

   (n) $\int_{-\pi/2}^{0} \frac{ds}{\cos s - 1}$

   (o) $\int_0^1 \frac{10 dx}{(x-1)^2 (x+2)}$

   (p) $\int_0^{\pi/2} \tan^3 x dx$

   (q) $\int_2^4 \frac{\sqrt{x^2 - 4}}{x} dx$

   (r) $\int_0^{\pi/2} \cos x \sin x dx$

   (s) $\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2} dy$

2. Determine if the following improper integrals converge or diverge.

   (a) $\int_0^2 \frac{dx}{1 - x}$

   (b) $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

   (c) $\int_0^{\pi/2} \cot \theta d\theta$

   (d) $\int_0^{\infty} \frac{ds}{1 + e^s}$

3. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in the domain of f, and f is called *odd* if $f(-x) = -f(x)$ for all x in the domain of f. A function which is even is symmetric about the y-axis, whereas a function which is odd is symmetric about the origin (the x-axis and y-axis).

   This symmetry is nice when we evaluate integrals over intervals of the form $(-a, a)$ where a is a real number or infinity:

$$\int_{-a}^{a} f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f \text{ is even,} \tag{1}$$

   and

$$\int_{-a}^{a} f(x) dx = 0 \quad \text{if } f \text{ is odd.} \tag{2}$$

   (a) Prove (2) by emulating the proof of (1) on p.297.

(b) Complete the table.

f	g	$f \circ g$
even	even	
even	odd	
odd	even	
odd	odd	

(c) Show that $\displaystyle\int_{-a}^{a} \sin^k x dx = 0$ whenever $k \geq 1$ is odd.

4. Let $y = 1/x$ on $[1, \infty)$. After revolving this curve about the x-axis the resulting surface is called *Gabriel's Horn*. The paradox of Gabriel's Horn says that we can fill the horn with a finite amount of paint, but to paint its surface we need an infinite amount of paint!

   (a) Show that the volume (p.316) of the horn is finite.

   (b) Show that the surface area (p.338) of the horn is infinite. [*hint:* use a comparison test.]

5. The *gamma* function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx.$$

   In this problem we will show that $\Gamma(n)$ converges for all $n > 0$.

   (a) Show that $\lim_{x\to\infty} x^a e^{-x} = 0$ for $a > 0$. [*hint*: plug in actual numbers for a (e.g. 2.1, 3.2, etc.) and see how many times you have to apply L'Hôpital's rule to get 0. Do you see a pattern?]

   (b) Show that for every positive integer n we can find a number M such that

$$0 < \frac{x^{n-1}}{e^x} \leq \frac{1}{x^2} \quad \text{for } x \geq M.$$

   [*hint*: use the previous part to show that $x^{n-1}e^{-x} = o(x^{-2})$ and then use the definition of the limit (p.71) to get M.]

   (c) Use part (b) and a comparison test to show that

$$\int_1^\infty x^{n-1} e^{-x} dx$$

   converges.

   (d) Use a comparison test to show that

$$\int_0^1 x^{n-1} e^{-x} dx$$

   converges. [*hint*: what is the largest value that e^{-x} takes on the interval $[0, 1]$?]

   (e) Conclude from (c) and (d) that $\Gamma(n)$ converges for each n.

6. Let's compute some values of the gamma function.

(a) Show that $\Gamma(1) = 1$.

(b) Use integration by parts to show that $\Gamma(n+1) = n\Gamma(n)$.

(c) Deduce from the previous two parts that $\Gamma(n+1) = n!$ whenever n is a positive integer.

(d) The function e^{-x^2} is called the *Gaussian*. We do not have the tools yet to compute the integral of the Gaussian over the real line, but here is its value:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Use the above equation and problem (3) to show that $\Gamma(1/2) = \sqrt{\pi}$.