(1.e) $a_n = e^{-n^2}$

$a_1 = e^{-1}$, $a_2 = e^{-4}$, $a_3 = e^{-9}$, $a_4 = e^{-16}$,

$a_5 = e^{-25}$

This sequence converges to 0.

Thm 5.4 says $\lim_{n\to\infty} \left(\frac{1}{e}\right)^n = 0$. Then

$$\lim_{n\to\infty} e^{-n^2} = \lim_{n\to\infty} \left(\frac{1}{e}\right)^{n^2}$$

$$= \underbrace{\lim_{n\to\infty} \left(\frac{1}{e}\right)^n \cdots \lim_{n\to\infty} \left(\frac{1}{e}\right)^n}_{\text{product n n-times}}$$

$$= 0$$

(1.h) $a_n = \left((-1)^n + 1\right)\left(\frac{n}{n+1}\right)$

$a_1 = 0$ $\quad a_3 = 0$ $\quad a_5 = 0$

$a_2 = 2 \cdot \frac{2}{3}$ $\quad a_4 = 2 \cdot \frac{4}{5}$ $\quad a_6 = 2 \cdot \frac{6}{7}$

This sequence diverges. The limit doesn't exist as the even terms converge to 2, but the odd terms converge to 0.

Slightly more rigorous. Suppose the sequence converges to L. Let's derive a contradiction. Let $\epsilon = 1$, then there is an N s.t.

$$|L - a_n| < 1 \quad \text{if } n > N$$

Since 0 occurs at every odd term, then

$$-1 < L < 1.$$

But the smallest even term is $4/3 > 1$. So that no even term lies in the ϵ-strip around L.

(1.i) $a_n = \ln(n+1) - \ln(n)$

$$= \ln\left(\frac{n+1}{n}\right)$$

$a_1 = \ln 2$ $\quad$ $a_3 = \ln 4/3$ $\quad$ $a_5 = \ln 6/5$

$a_2 = \ln 3/2$ $\quad$ $a_4 = \ln 5/3$

The sequence $\{a_n\}$ converges. Let $f(x) = \ln x$. f is continuous. And we know the sequence

$$\frac{n+1}{n} = 1 + \frac{1}{n} \longrightarrow 1$$

so that by thm 3 we get

$$\ln\left(\frac{n+1}{n}\right) = f\left(\frac{n+1}{n}\right) \to f(1) = \ln(1)$$
$$= 0.$$

(1.K) $a_n = (2n)^{1/2n}$

$a_1 = 2^{1/2}$ $\quad a_3 = 6^{1/6}$ $\quad a_5 = 10^{1/10}$

$a_2 = 4^{1/4}$ $\quad a_4 = 8^{1/8}$

This sequence converges. Note

$$(2n)^{1/2n} = \left((2n)^{1/n}\right)^{1/2}$$

$$= \left(2^{1/n} \cdot n^{1/n}\right)^{1/2}$$

Theorem 5 says $2^{1/n} \to 1$ and $n^{1/n} \to 1$

so that $2^{1/n} \cdot n^{1/n} \to 1 \cdot 1 = 1.$

Let $f(x) = x^{1/2}$. f is continuous so

theorem 3 says

$$f(2^{1/n} \cdot n^{1/n}) = (2^{1/n} \cdot n^{1/n})^{1/2}$$

$$= (2n)^{1/2n} \longrightarrow f(1) = 1^{1/2} = 1.$$

(1.n) $a_n = e^{-n}\cos n$

$a_1 = e^{-1}\cos 1$ $\quad$ $a_3 = e^{-3}\cos 3$ $\quad$ $a_5 = e^{-5}\cos 5$

$a_2 = e^{-2}\cos 2$ $\quad$ $a_4 = e^{-4}\cos 4$

This sequence converges.

Note: $-\frac{1}{e^n} \to 0$ and $\frac{1}{e^n} \to 0$.

We also know

$$-1 \leq \cos n \leq 1 \quad \text{for all } n.$$

So that

$$-\frac{1}{e^n} \leq \frac{\cos n}{e^n} \leq \frac{1}{e^n}$$

By the sandwich theorem $\frac{\cos n}{e^n} \to 0$.

(2.a) $\quad -1 \quad 8 \quad -27 \quad 64 \quad -125$

$\qquad\;\; 1 \quad\; 2 \quad\;\; 3 \quad\;\; 4 \quad\;\;\; 5$

$$\therefore \; a_n = (-1)^n n^3$$

(2.c) $\quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad 3$

$\qquad\;\; 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11$

$$\therefore \; a_n = \left\lfloor \frac{n}{3} \right\rfloor$$

$\lfloor \cdot \rfloor$ is the floor function.

(2.d) $\quad 2 \xrightarrow{+5} 7 \xrightarrow{+5} 12 \xrightarrow{+5} 17 \xrightarrow{+5} 22 \xrightarrow{+5} 27$

$\qquad\;\; 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6$

$$\therefore \; a_n = 5n - 3$$

(3.d) Bounded with positive terms but doesn't converge.

Soln: $\{a_n\} = \{1, 2, 1, 2, 1, 2, 1, 2, \ldots\}$

(3.e) Converges, has a constant subsequence, and a strictly increasing subsequence

Soln: Since it converges and has a constant subsequence, then it must converge to this constant. Start with

$$\{0, 0, 0, 0, 0, \ldots\}$$

Now "weave" in a strictly increasing sequence which converges to 0.

$$\{a_n\} = \{0, -1, 0, -1/2, 0, -1/3, 0, -1/4, 0, -1/5, \ldots\}$$

(3.h) Monotonic and bounded, but doesn't converge.

Soln: This is impossible since ~~[illegible]~~ every bounded, monotonic sequence converges.

4. $$a_{n+1} = \begin{cases} 3a_n+1 & a_n \text{ is odd} \\ \frac{1}{2}a_n & a_n \text{ is even} \end{cases}$$

(a) $a_1 = 11$ odd

$a_2 = 3a_1 + 1 = 3 \cdot 11 + 1 = 34$ even

$a_3 = \frac{1}{2} \cdot a_2 = \frac{1}{2} \cdot 34 = 17$ odd

$a_4 = 3 \cdot a_3 + 1 = 3 \cdot 17 + 1 = 52$ even

$a_5 = \frac{1}{2} a_4 = \frac{1}{2} \cdot 52 = 26$ even

$a_6 = \frac{1}{2} \cdot a_5 = \frac{1}{2} \cdot 26 = 13$ odd

$a_7 = 3 \cdot a_6 + 1 = 3 \cdot 13 + 1 = 40$ even

$a_8 = \frac{1}{2} \cdot a_7 = \frac{1}{2} \cdot 40 = 20$ even

$a_9 = \frac{1}{2} a_8 = \frac{1}{2} \cdot 20 = 10$ even

$a_{10} = \frac{1}{2} a_9 = \frac{1}{2} \cdot 10 = 5$ odd

$a_{11} = 3a_{10} + 1 = 3 \cdot 5 + 1 = 16$ even

$a_{12} = \frac{1}{2} a_{11} = \frac{1}{2} \cdot 16 = 8$ even

$a_{13} = \frac{1}{2} \cdot a_{12} = \frac{1}{2} \cdot 8 = 4$ even

$a_{14} = \frac{1}{2} \cdot a_{13} = \frac{1}{2} \cdot 4 = 2$ even

$a_{15} = \frac{1}{2} \cdot a_{14} = \frac{1}{2} \cdot 2 = 1$ odd

loop.

$a_{16} = \cancel{\frac{1}{2}}\, 3 \cdot a_{15} + 1 = 3 \cdot 1 + 1 = 4$

$a_{17} = 2$

$a_{18} = 1$

$a_{19} = 4$

$a_{20} = 2$

$a_{21} = 1$

$a_{22} = 4$

$a_{23} = 2$

$a_{24} = 1$

$a_{25} = 4$

$a_{26} = 2$

$a_{27} = 1$

$a_{28} = 4$

$\vdots$

(4.b) $a_1 = 25$

$\{a_n\} = \{25, 76, 38, 19, 58, 29,$

$88, 44, 22, 11,$

a_1 from last part

So from this point on

it will agree with

the sequence from part (a).

So eventually start looping on

4, 2, 1.

$a_1 = 21$

$\{a_n\} = \{21, 64, 32, 16, 8, 4, 2, 1, 4,$

$2, 1, 4, 2, 1, 4, 2, 1, \ldots\}$

(4c) The sequence eventually loops on

4, 2, 1. This is called the collatz

Conjecture. It hasn't been proved

that this happens for any a_1 !!!

(5.a)

(i) Let $a_n \to a$. Then

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty}\left(a_n^4 - 10a_n^2 - a_n + 9\right)$$

$$= \lim_{n\to\infty} a_n^4 - \lim_{n\to\infty} 10a_n^2 - \lim_{n\to\infty} a_n + \lim_{n\to\infty} 9$$

$$= \left(\lim_{n\to\infty} a_n\right)^4 - 10\left(\lim_{n\to\infty} a_n\right)^2 - \lim_{n\to\infty} a_n + \lim_{n\to\infty} 9$$

$$= a^4 - 10a^2 - a + 9$$

$$= f(a)$$

$\therefore$ f is cont. at a.

(iii) $f(x) = \left| \frac{\ln x}{x} \right|$

Let $a_n \to a$ we want to show $f(a_n) \to f(a)$.

Apply thm 3 twice working from the inside outward.

- $\ln$ is continuous so that
  $a_n \to a \implies \ln a_n \to \ln a$

- The quotient rule says
  $$\frac{\ln a_n}{a_n} \to \frac{\ln a}{a} \quad (*)$$

- Continuity of the absolute value $|\cdot|$ and $(*)$ then imply

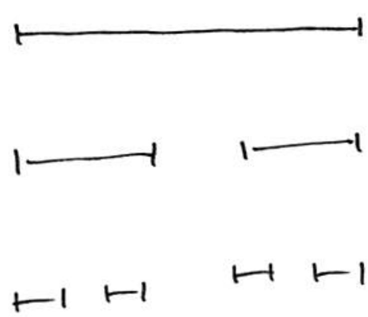
$$\lim_{n\to\infty} \left| \frac{\ln a_n}{a_n} \right| = \left| \frac{\ln a}{a} \right| = f(a).$$

(6.a)

length C_n = sum of the length of each piece.

at each level the length of each piece is the same

length C_n = (# pieces) $\cdot$ (length of a piece ~~length of each piece~~)

		# pieces	length of piece
C_0		$1 = 2^0$	$1 = 1/3^0$
C_1		~~#~~ $2 = 2^1$	$1/3 = 1/3^1$
C_2		$4 = 2^2$	$1/3 \cdot 1/3 = 1/9 = 1/3^2$
$\vdots$			
C_n		2^n	$1/3^n$

$\therefore$ length $C_n = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$

(6.b) $\therefore$ length $C = \lim_{n\to\infty}$ (length C_n) $= \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$