Math 242 Homework 4: due 7/8

1. List the first 5 terms of the sequence $\{a_n\}$. Write exact numbers like e^{-2}, $\ln 2$, etc. Determine if the sequence converges or diverges. If it converges, find its limit.

   (a) $a_n = n\pi \sin(n\pi/2)$
   (b) $a_n = \dfrac{n+2}{n^2+9n-1}$
   (c) $a_n = (-1)^n\left(1-\dfrac{1}{n}\right)$
   (d) $a_n = \left(1-\dfrac{1}{n}\right)^n$
   (e) $a_n = e^{-n^2}$
   (f) $a_n = \dfrac{1}{n}\displaystyle\int_1^n \dfrac{dx}{x}$
   (g) $a_n = n^{1/n}$
   (h) $a_n = ((-1)^n+1)\left(\dfrac{n}{n+1}\right)$
   (i) $a_n = \ln(n+1) - \ln n$
   (j) $a_n = \dfrac{n^2+1}{n^2}$
   (k) $a_n = (2n)^{1/2n}$
   (l) $a_n = \dfrac{(-\pi)^n}{5^n}$
   (m) $a_n = 2 + (0.2)^n$
   (n) $a_n = e^{-n}\cos(n)$
   (o) $a_n = \dfrac{\cos(n\pi)}{n}$
   (p) $a_n = \left(\dfrac{1}{5}\right)^n + 3^{n/2}$

2. Find a formula or in a few sentences describe what the following lists of numbers do.

   (a) $-1, 8, -27, 64, -125, \dots$
   (b) $0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \dots$
   (c) $0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, \dots$
   (d) $2, 7, 12, 17, 22, 27, 32, \dots$
   (e) $-9, -1, 7, 15, 23, 31, 39, 47, \dots$
   (f) $1, 2, 6, 24, 120, 720, 5040, \dots$
   (g) $1, 3, 6, 10, 15, 21, 28, 36, \dots$
   (h) $1, 5, 14, 30, 55, 91 \dots$
   (i) $1, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 3, 1, 2, 2, \dots$

3. Give an example of sequence $\{a_n\}$ which satisfies the given conditions. Provide a formula, a list of the terms, or draw a picture that describes what is happening in your sequence. You can use the same sequence for more than one answer. Exactly one of them does not exist.

   Definition: Let $\{a_n\}$ be a sequence and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of positive integers, then the sequence $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$ is called a *subsequence* of $\{a_n\}$.

   (a) Converges to π.
   (b) Converges to π, but none of the terms is π.
   (c) Converges to π and every term is rational. [*Hint*: use the decimal expansion of π.]
   (d) Bounded with positive terms, but does not converge.
   (e) Converges, has a constant subsequence, and a strictly increasing subsequence.
   (f) Converges and can be interpolated by a function of the form $f(x)\cos(\pi x)$.
   (g) Contains every integer in $\mathbb{Z}$.
   (h) Monotonic and bounded, but does not converge.
   (i) Has 3 constant subsequences.
   (j) Has m constant subsequences where m is a positive integer.

(k) Has infinitely many constant subsequences.

(l) A sequence with subsequences that can converge to any real number. The following two facts may be useful:

- There is a sequence $\{q_n\}$ which contains every rational.
- Every real number has a sequence of rationals converging to it.

4. Define the sequence $\{a_n\}$ recursively as follows

$$a_{n+1} = \begin{cases} 3a_n + 1 & \text{if } a_n \text{ is odd,} \\ \frac{1}{2}a_n & \text{if } a_n \text{ is even.} \end{cases}$$

(a) Find the first 40 terms of the sequence when $a_1 = 11$.

(b) Repeat the above exercise with $a_1 = 25$ and $a_1 = 21$.

(c) Keep repeating the above exercises varying a_1 until you see a pattern. What do you think happens to a sequence of this type?

5. There is a characterization of continuity in terms of sequences. Let $f : D \to \mathbb{R}$ be a function and let $a \in D$. Then the following two points are equivalent.

- If $\{a_n\}$ is a sequence in D converging to a, then the sequence $\{f(a_n)\}$ converges to $f(a)$. In symbols,
$$\lim_{n\to\infty} a_n = a \quad \Rightarrow \quad \lim_{n\to\infty} f(a_n) = f(a).$$
- f is continuous at a.

Let's show that $f(x) = x^3 + x - 1$ is continuous. Let a be any real number and let a_n be a sequence converging to a. Then

$$\begin{aligned} \lim_{n\to\infty} f(a_n) &= \lim_{n\to\infty} (a_n^3 + a_n - 1) \\ &= \lim_{n\to\infty} a_n^3 + \lim_{n\to\infty} a_n - \lim_{n\to\infty} 1 \\ &= (\lim_{n\to\infty} a_n)^3 + \lim_{n\to\infty} a_n - \lim_{n\to\infty} 1 \\ &= a^3 + a - 1 \\ &= f(a). \end{aligned}$$

Hence, f is continuous at a.

(a) Show that the following functions are continuous on their domains using the above method and the techniques we learned in section 9.1.

i. $f(x) = x^4 - 10x^2 - x + 9$

ii. $f(x) = \dfrac{3x^2 - 4x - 1}{x^3 - 2}$

iii. $f(x) = \left|\dfrac{\ln x}{x}\right|$. [*Hint*: use the theorem 3 in 9.1 (twice) with the continuity of the logarithm and absolute value.]

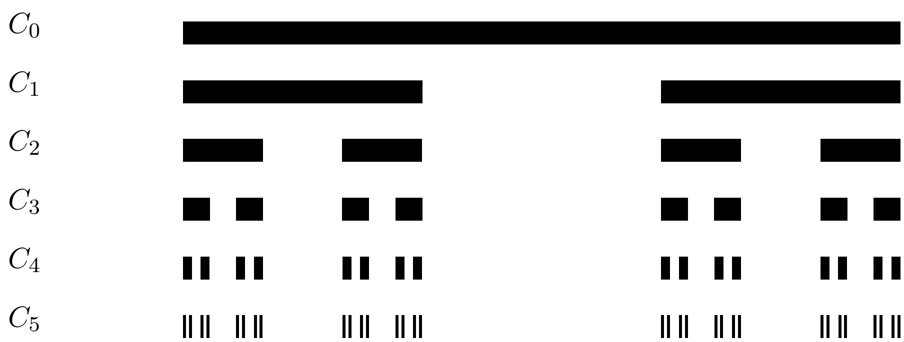
iv. $f(x) = \displaystyle\int_1^\infty \frac{dt}{t^x}$ for $x > 1$. Evaluate the integral, or you may use the following fact: $\displaystyle\lim_{n\to\infty} \int_1^\infty \frac{1}{t^{a_n} dt} = \int_1^\infty \lim_{n\to\infty} \frac{1}{t^{a_n}} dt.$

(b) Show that $f(x) = 1/x$ is discontinuous at $x = 0$. [*Hint*: do a proof by contradiction. Pick your favorite sequence of nonzero numbers $\{a_n\}$ that converge to 0, assume f is continuous so that the sequence $\{f(a_n)\}$ converges to some value. Derive a contradiction from this.]

6. The *Cantor set* C is constructed as follows: Take the unit interval $C_0 = [0,1]$ and remove the open middle third $(1/3, 2/3)$. Then the remaining set is $C_1 = [0,1/3] \cup [2/3,1]$. Now remove the open middle third from each of the remaining pieces. This leaves us with 4 pieces,

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Again, we remove the middle third from each which leaves us with 8 pieces. Continue this process indefinitely and what is left is C. Here is a visual of what is happening.



Some cool properties of the Cantor set:

- C contains no open intervals (a, b). So C is like a haze of points.
- C contains infinitely many numbers. In fact, as many as there are real numbers!
- The length of C is 0.

How can something have no length yet contain as many points as $(-\infty, \infty)$?!

(a) What is the length of each C_n?

(b) Deduce from the previous part that C has length 0.