9.2

(14) $$\sum_{n=0}^{\infty}\left(\frac{2^{n+1}}{5^n}\right) = \sum_{n=0}^{\infty} 2\cdot\frac{2^n}{5^n}$$

$$= \sum_{n=0}^{\infty} 2\cdot\left(\frac{2}{5}\right)^n$$

$$= \frac{2}{1-2/5} = \frac{10}{5-2} = \frac{10}{3}$$

(22) $$\sum_{n=1}^{\infty}\left(\tan^{-1}(n)-\tan^{-1}(n+1)\right)$$ telescoping

$$S_n = \left(\tan^{-1}(1) - \cancel{\tan^{-1}(2)}\right) + \left(\cancel{\tan^{-1}(2)} - \cancel{\tan^{-1}(3)}\right) + \cdots + \left(\cancel{\tan^{-1}(n-1)} - \tan^{-1}(n)\right)$$

$$= \tan^{-1}(1) - \tan^{-1}(n) = \pi/4 - \tan^{-1}(n)$$

$$\therefore \sum_{n=1}^{\infty}\left(\tan^{-1} n - \tan^{-1}(n+1)\right) = \lim_{n\to\infty}\left(\pi/4 - \tan^{-1}(n)\right) = \pi/4 - \pi/2$$

$$= -\pi/4$$

(38) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

$$\lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{2n+1}\right) = \ln\left(\tfrac{1}{2}\right) \neq 0$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$ diverges by the n^{th} term test

(54) $$0.\overline{d} = \frac{d}{10} + \frac{d}{100} + \frac{d}{1000} + \cdots$$

$$= \frac{d}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{d}{10}\left(\frac{1}{10}\right)^n$$

$$= \frac{d/10}{1 - 1/10}$$

$$= \frac{d}{10-1}$$

$$= \frac{d}{9}$$

9.3

(2). $\sum_{n=1}^{\infty} e^{-n}$. Let $f(x) = e^{-x}$

- f is cont.
- $f(x) = \frac{1}{e^x} > 0$ for all x.
- $f'(x) = -e^{-x} = -\frac{1}{e^x} < 0$ for all x

So f is decreasing

- $f(n) = e^{-n}$.

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} -\frac{1}{e^b} + \frac{1}{e^1}$$

$$= \frac{1}{e} \quad \text{converges.}$$

$\therefore$ By the integral test $\sum_{n=1}^{\infty} e^{-n}$ converges.

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $\left(p = 3/2 > 1\right)$,

then $\displaystyle -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

(10) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$

- $f(x) > 0$ for all x
- $f(n) = \dfrac{\ln n}{\sqrt{n}}$ $\forall n$
- f is continuous as a quotient of continuous functions.
- f is decreasing as $\longrightarrow$

$$f'(x) = \frac{d}{dx}\left(x^{-1/2}\ln x\right)$$

$$= x^{-1/2}\cdot\frac{1}{x} + \left(-\tfrac{1}{2}x^{-3/2}\right)\cdot\ln x$$

$$= \frac{2-\ln x}{x^{3/2}} < 0 \quad \text{for all } x>0.$$

Note

$$\int f(x)\,dx = \int \frac{\ln x}{\sqrt{x}}\,dx \qquad u=\ln x \quad dv = \frac{1}{\sqrt{x}}\,dx$$

$$du = \frac{1}{x}\,dx \quad v = 2\sqrt{x}$$

$$= 2\sqrt{x}\ln x - \int (2\sqrt{x})\cdot\left(\frac{1}{x}\,dx\right)$$

$$= 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx$$

$$= 2\sqrt{x}\cdot\ln x - 4\sqrt{x} + C$$

So that

$$\int_2^{\infty} f(x)\,dx = \lim_{b\to\infty} \left[2\sqrt{x}\cdot \ln x - 4\sqrt{x} \right]_2^b$$

$$= \lim_{b\to\infty} \underbrace{2\sqrt{b}\,(\ln b - 2)}_{\infty\cdot\infty} - \left(2\sqrt{2}\ln 2 - 4\sqrt{2}\right)$$

$$= \infty$$

Since $\int_2^{\infty} f(x)\,dx$ diverges, then by the integral test the series $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges.

(26) $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$. Let $f(x) = \frac{2}{1+e^n}$

Then it's not too hard to see that f is continuous, positive, decreasing function which interpolates the terms in the series.

Claim: $\int_1^{\infty} f(x)\,dx$ converges.

Let $g(x) = \frac{1}{e^x}$. In problem (2) we showed $\int_1^{\infty} g(x)\,dx$ converges. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{2}{1+e^x} \cdot \frac{e^x}{1}$$

$$= \lim_{x\to\infty} \frac{2}{1/e^x + 1} = 2$$

$\therefore$ LCT for improper integrals says $\int_1^{\infty} f(x)\,dx$ converges.

$\therefore$ The integral test says the series $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ converges.

9.4

$$(2) \quad \sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}} = \sum a_n$$

Let $b_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3}{n+\sqrt{n}} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{3}{1 + 1/\sqrt{n}} = 3$$

Since $\sum b_n$ diverges then by the LCT $\sum a_n$ diverges.

(10) $$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$$

$$\ln n < n^{c} \qquad \forall n \geq N = N(c)$$

$$\Longrightarrow (\ln n)^2 < n^{2c}$$

$$\Longrightarrow \frac{1}{n^{2c}} < \frac{1}{(\ln n)^2}$$

Take $c = 1/2$. Thus by the DCT

$\sum_{n=2}^{\infty} a_n$ diverges as $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

(14) $$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}} = \sum a_n$$

<u>Claim</u>: There is $p > 1$ s.t. $\frac{(\ln n)^2}{n^{3/2}} < \frac{1}{n^p}$

$\forall n \geq N = N(p)$.

For any $c > 0$ $\exists N = N(c)$ s.t.

$$\ln n < n^c \quad \forall n \geq N$$

$$\Rightarrow (\ln n)^2 < n^{2c}$$

$$\Rightarrow \frac{(\ln n)^2}{n^{3/2}} < \frac{n^{2c}}{n^{3/2}}$$

$$\Rightarrow \frac{(\ln n)^2}{n^{3/2}} < \frac{1}{n^{3/2 - 2c}}$$

So we just need to pick c so that

$$3/2 - 2c > 1$$

$$\Longrightarrow \quad -2c > 1 - 3/2 = -1/2$$

$$\Longrightarrow \quad c < 1/4$$

So picking $c < 1/4$ should work.

take $c = 1/5$. Then

$$3/2 - 2 \cdot c = 3/2 - 2(1/5)$$

$$= \frac{3}{2} - \frac{2}{5}$$

$$= \frac{15 - 4}{10}$$

$$= 11/10 > 1$$

Since $p = 11/10 > 1$ by the DCT

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}} \text{ converges.}$$

(26) $\sum_{n=1}^{\infty} \tan \frac{1}{n}$

Do limit comparison with the harmonic

$$\lim_{n\to\infty} \frac{\tan \frac{1}{n}}{1/n} = \lim_{h\to 0^+} \frac{\tan h}{h}$$

Let $n = \frac{1}{h}$

$$\overset{H}{=} \lim_{h\to 0^+} \frac{\sec^2 h}{1}$$

$$= 1$$

$\therefore$ Since the harmonic series diverges then by LCT the series $\sum_{n=1}^{\infty} \tan \frac{1}{n}$ diverges.

(3) Pizza Problem

$a_1 = 1/6$ of the pizza

$a_2 = 1/6 \cdot 1/6$ of the pizza

$a_3 = 1/6\,(1/6 \cdot 1/6)$ of the pizza and so on.

So $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n - 1$$

$$= \frac{1}{1 - 1/6} - 1$$

$$= \frac{6}{6-1} - 1$$

$$= \frac{6-5}{5} = 1/5$$

(5) Let $\{a_n\}$ be s.t. $0 \le a_n \le 9$

Show that $\displaystyle\sum_{n=1}^{\infty} a_n \frac{1}{10^n}$ converges.

Note the series has nonnegative terms and that

$$S_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$$

$$= 0.a_1 a_2 \ldots a_n$$

$$\le 1$$

for all n.

So the sequence of partial sums $\{S_n\}$ is bounded above

$\therefore$ Bounded Sum test says the series converges

(7) Show $\sum_{n=0}^{\infty} e^{-n^2}$ converges. using the integral test.

Let $f(x) = e^{-x^2}$

- f is positive: $f(x) = \frac{1}{e^{x^2}} > 0$
- f is continuous: f is the composition of two continuous functions e^x and $-x^2$.
- $f(n) = e^{-n^2}$. (Interpolates).
- f is decreasing for $x > 0$

$$f'(x) = -2x\, e^{-x^2} = -\frac{2x}{e^{x^2}} < 0$$

for all $x > 0$.

Since $e^{-x^2} \leq e^{-x}$ for $x \geq 1$

and $\int_1^\infty e^{-x} dx$ converges (see problem/solution

2 in section 9.2), then by

DCT for improper integrals $\int_1^\infty e^{-x^2} dx$

Converges

$\therefore$ the series $\sum_{n=0}^{\infty} e^{-n^2}$ converges.