9.5

(2) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^2}{e^n}\right)^{1/n} = \frac{1}{e} \cdot \lim_{n\to\infty} \left(n^2\right)^{1/n}$$

$$= \frac{1}{e} \cdot \lim_{n\to\infty} \left(n^{1/n}\right)^2$$

$$= \frac{1}{e} \left(\lim_{n\to\infty} n^{1/n}\right)^2$$

$$= \frac{1}{e}$$

Since $\frac{1}{e} < 1$ by the root test

the series $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$ converges.

(14) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right)^n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right) = 0$$

Since $0 < 1$ then the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right)^n$ converges.

(22) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n!}{(n+1)\cdot(n+1)^n} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n$$

The limit $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$ can be computed by taking ln and using L'Hôpital. There is a slicker way.

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^{-1}$$

$$= \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1}$$

$$= \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^{-1}$$

$$= e^{-1}$$

Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$, then by the ratio test the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges.

(46) $$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} \qquad (p \text{ constant})$$

Root test: First note that

$$\lim_{n\to\infty} (\ln n)^{1/n} = 0.$$

$$\lim_{n\to\infty} \ln (\ln n)^{1/n} = \lim_{n\to\infty} \frac{\ln(\ln n)}{n}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{\ln n} \cdot \frac{1}{n}}{1}$$

$$= \lim_{n\to\infty} \frac{1}{n \cdot \ln n} = 0$$

$$\therefore \lim_{n\to\infty} (\ln n)^{1/n} = e^0 = 1.$$

Hence, by continuity

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{1}{(\ln n)^p}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{(\ln n)^{1/n}}\right)^p$$

$$= \left(\lim_{n\to\infty} \frac{1}{(\ln n)^{1/n}}\right)^p = 1^p = 1.$$

So the root test is inconclusive.

Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1}{(\ln(n+1))^p} \cdot \frac{(\ln(n))^p}{1} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\ln(n)}{\ln(n+1)} \right)^p$$

$$= \left(\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} \right)^p$$

$$\overset{H}{=} \left(\lim_{n\to\infty} \frac{n+1}{n} \right)^p$$

$$= (1)^p$$

$$= 1$$

So the ratio test is inconclusive.

<u>9.6</u>

(6) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

- $\frac{\ln n}{n} > 0$ for all n.

- Since $\frac{d}{dx} \frac{\ln x}{x} < 0$ for $x > 0$

  then the sequence is decreasing

- $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{n} = 0.$

$\therefore$ AST says $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$ converges.

$\displaystyle\sum |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by DCT with $b_n = 1/n$.

(18) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$

Since $\frac{|\sin n|}{n^2} \le \frac{1}{n^2}$ for all n

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $(p=2)$

Then DCT says the series $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$

Converges absolutely, Hence, $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$

Converges. by ACT.

(30) $\sum_{n=1}^{\infty} (-5)^{-n} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{5}\right)^n$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ is a geometric series

with ratio $\frac{1}{5} < 1$ so converges.

Hence, $\sum_{n=1}^{\infty} (-5)^{-n}$ converges absolutely

so by ACT the series converges.

(34) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

$$\cos n\pi = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

$$= (-1)^n$$

S. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

Alternating Harmonic
Converges conditionally
proved in class/shown
in the text.

9.7

(2) $\sum_{n=0}^{\infty} (x+5)^n$

This power series converges for $|x+5| < 1$

or for $-6 < x < -4$.

$a = -5, \quad a + R = -4$

$\Longrightarrow \quad -5 + R = -4$

$\Longrightarrow \quad R = 1$

Interval of convergence: $(-6, -4)$

Radius of convergence: $R = 1$

Converges absolutely on $(-6, -4)$

converges conditionally nowhere.

(10) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

Apply n^{th} root test to $\sum |a_n|$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-1|}{(\sqrt{n})^{1/n}}$$

$$= \lim_{n\to\infty} \frac{|x-1|}{(n^{1/n})^{1/2}}$$

$$= |x-1|$$

So converg abs. on $|x-1| < 1$.

or for $0 < x < 2$.

Test endpoints

$x = 0$: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

- $\frac{1}{\sqrt{n}} > 0$ for all n.

- $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all n.

- $\frac{1}{\sqrt{n}} \to 0$

So AST says $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges.

But $\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

$(p = 1/2)$.

$x = 2$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $(p = 1/2)$

Interval of Convergence : $[0, 2)$

Radius of Convergence : $R = 1$

Converge absolutely on : $(0, 2)$

Converge conditionally at $x = 0$.

(14) $\displaystyle\sum_{n=0}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$

Apply Ratio test to $\displaystyle\sum_{n=0}^{\infty} |a_n|$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|2x+3|^{2n+3}}{(n+1)!} \cdot \frac{n!}{|2x+3|^{2n+1}}$$

$$= \lim_{n\to\infty} \frac{|2x+3|^2}{n+1}$$

$$= 0 \quad \text{for all } x.$$

Interval of convergence: $(-\infty, \infty)$

Radius of convergence: $R=\infty$

Converges absolutely everywhere

Converges conditionally nowhere

Additional Problems

(2.a) $$y = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$y(0) = 1 + 0 + \frac{0^2}{2!} + \frac{0^3}{3!} + \cdots = 1$$

$$\frac{dy}{dx} = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^n}{n!} = \frac{d}{dx} \cdot 1 + \frac{d}{dx} \cdot x + \frac{d}{dx} \frac{x^2}{2!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = 0 + 1 + \frac{2x}{2!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(3) $$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$y' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$y'' = 0 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$$

$$y'' + y''' = (-x + x) + \left(\frac{x^3}{3!} - \frac{x^3}{3!}\right) + \left(-\frac{x^5}{5!} + \frac{x^5}{5!}\right) + \cdots$$

$$= 0.$$

(4.b) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ so $\pi = 3\tan^{-1}\sqrt{3}$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\Longrightarrow \quad \pi = 3\tan^{-1}\sqrt{3}$$

$$= 3 \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{3})^{2n+1}}{2n+1} = 3\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2n+1}$$

(5.a) Let $f(x) = \frac{1}{1-x}$

then $f(x) = 1 + x + x^2 + x^3 + \cdots$ for $1 < x < 1$

$\Rightarrow f'(x) = 0 + 1 + 2x + 3x^2 + \cdots$

$= \frac{1}{(1-x)^2}$

$\therefore f'(-x) = \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

so $f'(-1) = \frac{1}{(1+1)^2} = 1 - 2 + 3 - 4 + 5 - 6 + \cdots$