9.8

(2) $f(x) = \ln(1+x), \quad a = 0$

$f'(x) = \dfrac{1}{1+x} \Longrightarrow f'(0) = 1$

$f''(x) = -\dfrac{1}{(1+x)^2} \Longrightarrow f''(0) = -1$

$f'''(x) = \dfrac{2}{(1+x)^3} \longrightarrow f'''(0) = 2$

$P_0(x) = f(a) = 0$

$P_1(x) = P_0(x) + f'(a)x = x$

$P_2(x) = P_1(x) + \dfrac{f''(a)}{2!}x^2 = x - \frac{1}{2}x^2$

$P_3(x) = P_2(x) + \dfrac{f'''(a)}{3!}x^3 = x - \frac{1}{2}x^2 + \frac{2}{6}x^3$

(4) $f(x) = \frac{1}{x+2}, \quad a = 0$

$$f'(x) = -\frac{1}{(x+2)^2} \Rightarrow f'(0) = -\frac{1}{4}$$

$$f''(x) = \frac{2\cdot 1}{(x+2)^3} \Rightarrow f''(0) = \frac{2!}{8}$$

$$f'''(x) = -\frac{3\cdot 2\cdot 1}{(x+2)^4} \Rightarrow f'''(0) = -\frac{3!}{16}$$

$$P_0(x) = f(a) = \frac{1}{2}$$

$$P_1(x) = P_0(x) + f'(a)x = \tfrac{1}{2} - \tfrac{1}{4}x$$

$$P_2(x) = P_1(x) + \frac{f''(a)}{2!}x^2 = \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2$$

$$P_3(x) = P_2(x) + \frac{f'''(a)}{3!}x^3 = \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3$$

(12) $f(x) = \frac{1}{1-x}$, $a = 0$

$f'(x) = \frac{1}{(1-x)^2}$ $\qquad$ $f''(x) = \frac{2 \cdot 1}{(1-x)^3}$

$f'''(x) = \frac{3 \cdot 2 \cdot 1}{(1-x)^4}$

$\longrightarrow \quad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

$\longrightarrow \quad f^{(n)}(0) = n!$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} x^n$$

(20) $f(x) = (x+1)^2, \quad a = 0$

$f'(x) = 2(x+1)^1 \longrightarrow f'(0) = 2$

$f''(x) = 2\cdot 1\,(x+1)^0 = 2 \longrightarrow f''(0) = 2$

$f'''(x) = 0$ so $f^{(n)}(x) = 0 \quad \forall n \geq 3.$

and $f^{(n)}(a) = 0 \quad \forall n \geq 3$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2$$

$$= 1 + 2x + x^2$$

(22) $f(x) = 2x^3 + x^2 + 3x - 8$; $a = 1$

$f'(x) = 6x^2 + 2x + 3 \Rightarrow f'(a) = 11$

$f''(x) = 12x + 2 \Rightarrow f''(a) = 14$

$f'''(x) = 12 \Rightarrow f'''(a) = 12$

$f^{(n)}(x) = 0 \ \forall n \geq 4 \Rightarrow f^{(n)}(a) = 0 \ \forall n \geq 4$

$$\sum_{n=0}^{\infty} \frac{f^{n}(a)}{2!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$$

(26) $f(x) = \dfrac{x}{1-x}, \quad a = 0$

Maclaurin Series for $\dfrac{1}{1-x}$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \quad \frac{x}{1-x} = x \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

(28) $f(x) = 2^x, \quad a = 1$

$f'(x) = 2^x \ln 2 \qquad f''(x) = 2^x (\ln 2)^2$

$f'''(x) = 2^x (\ln 2)^3$

$\Rightarrow f^{(n)}(x) = 2^x (\ln 2)^n$

$\Rightarrow f^{(n)}(1) = 2 (\ln 2)^n$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(x-1)^n$$

9.9

(2) $f(x) = e^{-x/2}$

$$= \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2^n \cdot n!} x^n$$

(4) $\sin\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{\pi}{2}x)^{2n+1}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{\pi}{2}\right)^{2n+1} \cdot \frac{1}{(2n+1)!} x^{2n+1}$$

(8) $$x^2 \sin x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!}$$

(12) $$x^2 \cos x^2 = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^2 \cdot x^{4n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{(2n)!}$$

(10) $\sin x - x + \dfrac{x^3}{3!}$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x + \frac{x^3}{3!}$$

$$= \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$= \sum_{n=2}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(14) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$$= \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!}$$

$$= \frac{1}{2} - \sum_{n=0}^{\infty}(-1)^n 2^{2n-1}\frac{x^{2n}}{(2n)!}$$

(30) $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \cdots$$

$$\frac{e^x - e^{-x}}{x} = 2 + 2\frac{x^2}{3!} + 2\frac{x^4}{5!} + \cdots$$

$$\Rightarrow \lim_{x \to 0} \frac{e^x - e^{-x}}{x} = 2$$

Additional Problem

$$f^{(n)}(a) = \begin{cases} 0 & n \text{ is even} \\ \dfrac{e^{a}+e^{-a}}{2} & n \text{ is odd} \end{cases}$$

(a) If $x=3$, then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^{3}+e^{-3}}{2\cdot(2n+1)}(x-3)^{2n+1}$$

(b) If $x=0$, then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

(c) $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \frac{x^2}{(2n+3)(2n+2)}$$

$\therefore$ $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \quad \text{for all } x.$$

$\therefore$ The Maclaurin Series for f converges for all f $(-\infty, \infty)$.