MATH 242 Summer 2016
Practice Midterm 1

Name: ______________________________

Instructions:

- Begin by writing your name in the space provided above.
- You have 80 minutes to complete this exam.
- No phones, calculators, notes, or any form of assistance may be used during the exam.
- You must show all of your work, unless you are asked not to. Answers which are cryptic or have no supporting evidence will most likely not receive full credit. When in doubt, ask.
- Please be organized! Answer questions in the spaces provided as neatly as possible. If you run out of room, then continue on a piece of scratch paper and make clear note of it.

Question	Points	Score
1	10	
2	10	
3	25	
4	35	
5	10	
6	10	
7	10	
Total:	110	

1. (10 points) True/False. Circle your answer. You do not need to show work.

(a) True False If a function is differentiable and its derivative is always positive, then the derivative of its inverse is always positive.

(b) True False If $y = f(x)$ is a solution to a differential equation, then $y = f(x) + 1$ is also a solution to the same differential equation.

(c) True False The following calculation is correct

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

(d) True False Since $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ on the interval $[1,\infty)$ and $\int_1^\infty \frac{1}{x^2}dx$ converges, then $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ converges.

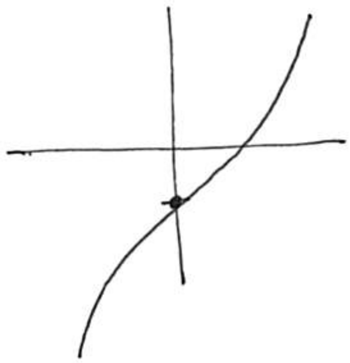
(e) True False A function can never be its own inverse.

2. Inverses.

(a) (2 points) Is the function $f(x) = x^3 - 1$ one-to-one? Justify your answer.

Yes.

Passes HLT



OR

If $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 - 1 = x_2^3 - 1$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\therefore f$ is 1-1

OR

$f'(x) = 3x^2$ is positive for all real numbers $x \neq 0$.

$\therefore f$ is increasing

$\therefore f$ is 1-1.

(b) (3 points) Find the inverse of $g(x) = \ln(x+1) - \ln x$.

$$g(x) = \ln(x+1) - \ln x = \ln\left(\frac{x+1}{x}\right)$$

Solve $g(y) = x$ for y.

$$\ln\left(\frac{y+1}{y}\right) = x$$

$$\frac{y+1}{y} = e^x$$

$$y - e^x y = -1$$

$$y(1 - e^x) = -1$$

$$y = \frac{1}{e^x - 1}$$

$$\therefore g^{-1}(x) = \frac{1}{e^x - 1}.$$

(c) (5 points) Let $h(x) = x^2 - 7x + 12$ for $x > 7/2$. Use the IFT to compute the derivative of h^{-1} at the point $x = h(4) = 0$.

$$h(4) = 0 \Longleftrightarrow h^{-1}(0) = 4$$

$$h'(x) = 2x - 7$$

$$(h^{-1})'(0) = \frac{1}{h'(h^{-1}(0))} = \frac{1}{h'(4)}$$

$$= \frac{1}{2 \cdot 4 - 7} = 1.$$

3. Derivatives. Find dy/dx.

(a) (5 points) $y = \cos^{-1}(x^\pi) + \tan^{-1}(x^\pi)$

$$y' = -\frac{1}{\sqrt{1-(x^\pi)^2}} \cdot \frac{d}{dx}\left(x^\pi\right) + \frac{1}{(x^\pi)^2+1} \cdot \frac{d}{dx}\left(x^\pi\right)$$

$$= -\frac{1}{\sqrt{1-x^{2\pi}}}\left(\pi (x)^{\pi-1}\right) + \frac{1}{x^{2\pi}+1}\left(\pi x^{\pi-1}\right)$$

(b) (5 points) $y = \log_2 \sqrt{1-x^2}$ $= \frac{1}{2}\log_2(1-x^2) = \frac{1}{2} \cdot \frac{\ln(1-x^2)}{\ln 2}$

$$y' = \frac{1}{2\ln 2} \frac{d}{dx}\left(\ln(1-x^2)\right)$$

$$= \frac{1}{2\ln 2} \cdot \frac{1}{1-x^2} \cdot \frac{d}{dx}\left(1-x^2\right) = \frac{-2x}{2\ln 2\,(1-x^2)}$$

$$= \frac{x}{\ln 2\,(x^2-1)}$$

(c) (5 points) $y = 2^{\ln x + \sqrt{x}}$

$$y' = 2^{\ln x + \sqrt{x}} \cdot \frac{d}{dx}\left(\ln x + \sqrt{x}\right) \cdot \ln 2$$

$$= 2^{\ln x + \sqrt{x}} \cdot \ln 2 \left(\frac{1}{x} + \frac{1}{2}\frac{1}{\sqrt{x}}\right)$$

(d) (5 points) $y = \dfrac{x^2 \cos x \sin x}{x-1}$ $\Rightarrow$ $\ln y = 2\ln x + \ln \cos x + \ln \sin x - \ln(x-1)$

$$\frac{y'}{y} = \frac{2}{x} + \frac{1}{\cos x}\cdot(-\sin x) + \frac{1}{\sin x}\cdot \cos x - \frac{1}{x-1}\cdot 1$$

$$y' = y\left(\frac{2}{x} - \tan x + \cot x - \frac{1}{x-1}\right)$$

$$= \frac{x^2 \cos x \sin x}{x-1}\left(\frac{2}{x} - \tan x + \cot x - \frac{1}{x-1}\right)$$

(e) (5 points) $y = \dfrac{e^{2x} - e^{-2x}}{2}$ $= \dfrac{e^{2x}}{2} - \dfrac{e^{-2x}}{2}$

$$y' = \frac{1}{2}e^{2x}\cdot 2 - \frac{1}{2}e^{-2}\cdot(-2)$$

$$= e^{2x} + e^{-2x}$$

4. Integration. Evaluate the following integrals

(a) (7 points) $\int_0^{2\pi} \frac{d\theta}{\cos^2\theta + \sin^2\theta}$

$$\int_0^{2\pi} \frac{d\theta}{\cos^2\theta + \sin^2\theta} = \int_0^{2\pi} d\theta = 2\pi$$

(b) (7 points) $\int_1^{\infty} \frac{\ln x}{x} dx$ $= \lim_{b\to\infty} \int_0^{\ln b} u\, du$

$u = \ln x$

$du = \frac{1}{x} dx$

$$= \lim_{b\to\infty} \left. \frac{u^2}{2} \right|_0^{\ln b}$$

$$= \lim_{b\to\infty} \frac{(\ln b)^2}{2} - 0$$

$$= \infty$$

(c) (7 points) $\int \ln x^2 dx$ $= 2\int \ln x \, dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$$= 2\left[x\ln x - \int (x)\left(\frac{1}{x}dx\right)\right]$$

$$= 2\left[x\ln x - \int dx\right]$$

$$= 2\left[x\ln x - x\right] + C.$$

(d) (7 points) $\int \frac{x dx}{\sqrt{x^2+4}}$ $= \int \frac{2\tan\theta\,(2\sec^2\theta\, d\theta)}{2\sec\theta}$

$x = 2\tan\theta$

$x^2 = 4\tan^2\theta$

$\sqrt{x^2+4} = \sqrt{4(\tan^2\theta+1)}$

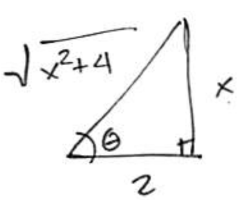
$= \sqrt{4\sec^2\theta}$

$= 2|\sec\theta|$

$= 2\sec\theta$

$(\sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2})$

$dx = 2\sec^2\theta\, d\theta$



$$= 2\int \tan\theta\sec\theta\, d\theta$$

$$= 2\sec\theta + C$$

$$= 2\left(\frac{\sqrt{x^2+4}}{2}\right) + C$$

$$= \sqrt{x^2+4} + C$$

(e) (7 points) $\int_2^3 \frac{dx}{x^2 - x} = \int_2^3 \frac{dx}{x(x-1)}$

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$1 = A(x-1) + Bx$$

$$= (A+B)x - A$$

$$\begin{cases} A + B = 0 \\ -A = 1 \end{cases} \Longrightarrow (-1) + B = 0$$

$$B = 1$$

$$\therefore \int_2^3 \frac{dx}{x(x-1)} = \int_2^3 -\frac{1}{x} + \frac{1}{x-1}\,dx$$

$$= -\ln|x| + \ln|x-1| \Big|_2^3$$

$$= -\ln 3 + \ln 2 - (-\ln 2 + \ln 1)$$

$$= 2\ln 2 - \ln 3$$

$$= \ln 4/3$$

$$x^2 = Cy - y^2$$
$$\Rightarrow x^2 - y^2 = Cy - 2y^2 \quad (*)$$

5. Differential Equations.

(a) (4 points) Let C be a constant. Show that $x^2 + y^2 = Cy$ is a solution to the differential equation $y' = \dfrac{2xy}{x^2 - y^2}$. Use implicit differentiation to find y'.

$$x^2 + y^2 = Cy$$
$$\Rightarrow 2x + 2y \cdot y' = Cy'$$
$$\cancel{2y y' = Cy' - 2x}$$
$$\cancel{y' = Cy' - 2x}$$
$$2x = Cy' - 2yy'$$
$$= y'(C - 2y)$$
$$\Rightarrow y' = \frac{2x}{C - 2y}$$

$$y' = \frac{2xy}{x^2 - y^2}$$
$$\frac{2x}{C - 2y} = \frac{2xy}{x^2 - y^2}$$
$$2x(x^2 - y^2) = 2xy(C - 2y)$$
$$2x(x^2 - y^2) = 2x(Cy - 2y^2)$$
$$= 2x(x^2 - y^2) \quad (*)$$

✓

(b) (6 points) Solve the differential equation $(x+1)y' = xe^{-y}$. Simplify your answer.

$$(x+1)\frac{dy}{dx} = x e^{-y}$$
$$\Rightarrow e^{y} dy = \frac{x}{x+1} dx$$
$$\Rightarrow \int e^{y} dy = \int \frac{x}{x+1} dx$$
$$e^{y} = \int \frac{x+1-1}{x+1} dx = \int dx - \int \frac{1}{x+1} dx$$
$$= x - \ln|x+1| + C$$
$$\Rightarrow y = \ln(x - \ln|x+1| + C)$$

6. (10 points) Exponential Change. Moore's Law says that the number of transistors on integrated circuits doubles every two years. The first circuit, created in 1965, had 10 transistors. After how many years does Moore's law predict that circuits will have 10 billion transistors? (Extra credit: approximate the *year*.)

$$y = y_0 e^{kt}$$

Given $y_0 = 10$ and $y(2) = 20$.

$$20 = y(2) = 10e^{k \cdot 2}$$

$$2 = e^{k \cdot 2}$$

$$\frac{\ln 2}{2} = k$$

$$\therefore y = 10e^{\frac{\ln 2}{2} \cdot t}$$

Want t s.t. $y(t) = 10 \times 10^9$

$$\cancel{10} \times 10^9 = \cancel{10} e^{\frac{\ln 2}{2} \cdot t}$$

$$10^9 = e^{\frac{\ln 2}{2} \cdot t}$$

$$\ln 10^9 = \frac{\ln 2}{2} \cdot t$$

$$t = 2 \frac{\ln 10^9}{\ln 2} \text{ years}$$

7. Limits. Compute the following limits.

(a) (5 points) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ Let $f(x) = \left(1+\frac{3}{x}\right)^x$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} x\cdot\ln\left(1+\frac{3}{x}\right) \qquad \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/x} \qquad \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+3/x}\cdot -3x^{-2}}{-x^{-2}}$$

$$= \lim_{x\to\infty} \frac{3}{1+3/x} = 3$$

$$\therefore \lim_{x\to\infty} f(x) = e^3$$

(b) (5 points) $\lim_{x\to 0}\frac{xe^x}{\sin x}$

$$\lim_{x\to 0^+} \frac{xe^x}{\sin x} \qquad \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x\to 0^+} \frac{xe^x+e^x}{\cos x}$$

$$= \frac{0\cdot 1+1}{1}$$

$$= 1$$

Similarly $\lim_{x\to 0^-} \frac{xe^x}{\sin x} = 1$

$$\therefore \lim_{x\to 0} \frac{xe^x}{\sin x} = 1$$