MATH 242 Summer 2016
Practice Midterm 2

Name: ____

Instructions:

- Begin by writing your name in the space above.
- You have 80 minutes to complete this exam.
- No phones, calculators, notes, or any form of assistance may be used during the exam.
- You must show all of your work, unless you are asked not to. Answers which are cryptic or have no supporting evidence will most likely not receive full credit. When in doubt, ask.
- Please be organized! Answer questions in the space provided as neatly as possible. If you run out of room, continue on a piece of scratch paper and make a clear note of it.

Question	Points	Score
1	10	
2	10	
3	10	
4	15	
5	50	
6	15	
Total:	110	

1. (10 points) True/False. Circle your answer. You do not need to show work.

(a) True False If f and g are invertible functions and the composition $f \circ g$ makes sense, then

$$(f \circ g)^{-1}(x) = (f^{-1} \circ g^{-1})(x).$$

(b) True False If f is differentiable and always positive, then

$$f'(x) = f(x)\frac{d}{dx}(\ln f(x)).$$

(c) True False The integral $\int_1^\infty \frac{dx}{x^{1.01}}$ converges.

(d) True False A function that is increasing or decreasing on $\mathbb{R}$ is invertible.

(e) True False An invertible function on $\mathbb{R}$ is either increasing or decreasing.

2. Inverses.

(a) (2 points) Give an example of a 1-1 function.

$f(x) = x$

$f(x) = x^2, \quad x > 0$

$f(x) = x^3$

$f(x) = e^x$

(b) (3 points) Find the inverse of $f(x) = x^2 + 2x$ for $x < -1$. (*)

Solve $f(y) = x$ for y.

$y^2 + 2y = x$

$y^2 + 2y - x = 0$

$a = 1$

$b = 2$

$c = -x$

$y = \dfrac{-2 \pm \sqrt{4 - 4(1)(-x)}}{2(1)}$

$= \dfrac{-2 \pm \sqrt{4(x+1)}}{2}$

$= -1 \pm \sqrt{x+1}$

(*) $y < -1$ so we take the minus

$f^{-1}(x) = -1 - \sqrt{x+1}$

(c) (5 points) Let $g(x) = x + \ln x$ for $x > 0$. Find $(g^{-1})'(g(e))$.

~~$g(e) = 2e + \ln e = e + 1$~~

$b = g(e) \iff g^{-1}(b) = e$

$g'(x) = 1 + \frac{1}{x}$

$(g^{-1})'(\underbrace{g(e)}_{b}) = \dfrac{1}{g'(g^{-1}(b))} = \dfrac{1}{g'(e)} = \dfrac{1}{1 + \frac{1}{e}}$

$= \dfrac{e}{e+1}$

3. Limits. Compute the following limits.

(a) (5 points) $\lim_{x\to 0^+} x^x$

Let $f(x) = x^x$

$$\Rightarrow \lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} x \ln x \qquad 0\cdot(-\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln x}{1/x} \qquad \frac{-\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} -x = 0$$

$$\therefore \lim_{x\to 0^+} f(x) = e^0 = 1.$$

(b) (5 points) $\lim_{x\to 1} \csc(\pi x)\ln x$

$$\lim_{x\to 1^+} \csc(\pi x)\ln x \qquad -\infty\cdot 0$$

$$= \lim_{x\to 1^+} \frac{\ln x}{\sin \pi x} \qquad \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{1/x}{\pi\cos(\pi x)}$$

$$= \lim_{x\to 1^+} \frac{1}{\pi x \cos(\pi x)}$$

$$= -\frac{1}{\pi}$$

$$\lim_{x\to 1^-} \csc(\pi x)\ln x \qquad +\infty\cdot 0$$

$\vdots$

$$= \lim_{x\to 1^-} \frac{1}{\pi x\cos(\pi x)}$$

$$= -\frac{1}{\pi}$$

$$\therefore \lim_{x\to 1} \csc(\pi x)\ln x = -\frac{1}{\pi}$$

4. Derivatives. Find dy/dx.

(a) (5 points) $y = \dfrac{(\sin^{-1} x)^4 (x^2+1)^7}{(2x+1)^3}$ $\Rightarrow \ln y = 4\ln(\sin^{-1} x) + 7\ln(x^2+1) - 3\ln(2x+1)$

$$\frac{y'}{y} = 4 \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + 7 \cdot \frac{1}{x^2+1} \cdot 2x - 3 \frac{1}{2x+1} \cdot 2$$

$$y' = y\left(\frac{4}{\sin^{-1}x\sqrt{1-x^2}} + \frac{14x}{x^2+1} - \frac{6}{2x+1}\right)$$

$$= \left(\frac{(\sin^{-1}x)^4 (x^2+1)^7}{(2x+1)^3}\right)\left(\frac{4}{\sin^{-1}x\sqrt{1-x^2}} + \frac{14x}{x^2+1} - \frac{6}{2x+1}\right)$$

(b) (5 points) $y = x^{x^e}$, for $x > 0$

$$\ln y = x^e \ln x$$

$$\frac{y'}{y} = e x^{e-1} \cdot \ln x + x^e \cdot \frac{1}{x}$$

$$y' = x^{x^e}\left(e x^{e-1} \cdot \ln x + x^{e-1}\right)$$

(c) (5 points) $y = \log_5(x) - (\cot^{-1}(x))^{-1}$ $= \dfrac{\ln x}{\ln 5} - (\cot^{-1}(x))^{-1}$

$$y' = \frac{1}{\ln 5} \cdot \frac{1}{x} + (\cot^{-1}(x))^{-2} \cdot \frac{d}{dx}(\cot^{-1} x)$$

$$= \frac{1}{\ln 5} \cdot \frac{1}{x} + (\cot^{-1} x)^{-2}\left(-\frac{1}{x^2+1}\right)$$

5. Integration. Evaluate the following integrals

(a) (10 points) $\int x \sec x \tan x dx$ $= I$

$u = x$ $\quad dv = \sec x \tan x$

$du = dx$ $\quad v = \sec x$

$$I = x \sec x - \int \sec x \, dx$$

$$= x \sec x - \ln|\sec x + \tan x| + C.$$

(b) (10 points) $\int x\sqrt{1-x^4}dx$ $= \frac{1}{2}\int \sqrt{1-u^2}\,du$

$u = x^2$

$du = 2x\,dx$

$u = \sin\theta$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta}$

$= \cos\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{2}\int (\cos\theta)(\cos\theta\, d\theta)$$

$$= \frac{1}{2}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int 1 + \cos 2\theta\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

(c) (10 points) $\int e^{\sqrt{x}}dx$

$$= 2\int e^u u\,du$$

$$\begin{array}{ccc} u & \searrow^{+} & e^u \\ 1 & \searrow^{-} & e^u \\ 0 & & e^u \end{array}$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$\Rightarrow dx = 2\sqrt{x}\,du$

$= 2u\,du$

$$= 2\left(ue^u - e^u\right) + C$$

$$= 2\left(\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}}\right) + C$$

(d) (10 points) $\int_0^1 \frac{dy}{4y-1}$

$$= \int_0^{1/4} \frac{dy}{4y-1} + \int_{1/4}^1 \frac{dy}{4y-1}$$

$u = 4y-1$

$du = 4dy$

$$= \frac{1}{4}\int_{-1}^{0} \frac{du}{u} + \frac{1}{4}\int_{0}^{3} \frac{du}{u}$$

$$= \frac{1}{4}\lim_{b\to 0^-}\int_{-1}^{b} \frac{du}{u} + \frac{1}{4}\lim_{a\to 0^+}\int_{a}^{3} \frac{du}{u}$$

$$= \frac{1}{4}\lim_{b\to 0^-} \underbrace{\ln|b|}_{\to\infty} - \frac{1}{4}\ln|-1| + \frac{1}{4}\ln|3| - \frac{1}{4}\lim_{a\to 0^+}\underbrace{\ln|a|}_{\to\infty}$$

~~[illegible]~~ limit d.n.e.

(e) (10 points) $\int \frac{x^2+1}{x(x^2+3)}dx$ $= \frac{1}{3}\int \frac{du}{u}$

$u = x^3 + 3x$ $= \frac{1}{3}\ln|u| + C$

$du = (3x^2+3)\,dx$

$= 3(x^2+1)\,dx$

$= \frac{1}{3}\ln|x^3+3x| + C.$

6. Differential equations and exponential change.

(a) (4 points) Verify that $y = C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x$ is a solution to the differential equation $y'' + y = \sin x$.

$$y' = -C_1 \sin x + C_2 \cos x - \frac{1}{2}\cos x + \frac{1}{2} x \sin x$$

$$y'' = -C_1 \cos x - C_2 \sin x + \frac{1}{2}\sin x + \frac{1}{2}\sin x + \frac{1}{2} x \cos x$$

$$= -C_1 \cos x - C_2 \sin x + \sin x + \frac{1}{2} x \cos x$$

$$\Rightarrow y'' + y = -C_1 \cos x - C_2 \sin x + \sin x + \frac{1}{2} x \cos x$$
$$+ C_1 \cos x + C_2 \sin x \qquad - \frac{1}{2} x \cos x$$

$$= \sin x$$

✓

(b) (6 points) Solve the differential equation $\frac{dL}{dt} - te^{t-L} = 0$.

$$\frac{dL}{dt} = te^{t-L}$$

$$\int e^{L}\, dL = \int te^{t}\, dt$$

$$e^{L} = e^{t}(t-1) + C$$

$$L = \ln\left(e^{t}(t-1) + C\right)$$

$t \quad + \quad e^t$

$1 \quad - \quad e^t$

$0 \quad \quad e^t$

(c) (5 points) If the chemical reaction $N_2O_5 \rightarrow 2NO_2 + \frac{1}{2}O_2$ occurs at 45°, then the rate of reaction of dinitrogen pentoxide is proportional to its concentration $[N_2O_5]$ as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5].$$

How long will the reaction take to reduce to half of its original value?

$K = 0.0005$

$$[H_2O_5] = [N_2O_5]_0 \, e^{0.0005 \cdot t}$$

want t so that $[N_2O_5](t) = \frac{1}{2}[N_2O_5]_0$

$$\frac{1}{2}\cancel{[N_2O_5]_0} = \cancel{[N_2O_5]_0} \, e^{0.0005 \cdot t}$$

$$\ln \tfrac{1}{2} = 0.0005\, t$$

$$t = \frac{\ln \frac{1}{2}}{0.0005}$$