MATH 242 Summer 2016
Practice Midterm 3

Name: ____________________

Instructions:

- Begin by writing your name in the space above.
- You have 80 minutes to complete this exam.
- No phones, calculators, notes, or any form of assistance may be used during the exam.
- You must show all of your work, unless you are asked not to. Answers which are cryptic or have no supporting evidence will most likely not receive full credit. When in doubt, ask.
- Please be organized! Answer questions in the space provided as neatly as possible. If you run out of room, continue on a piece of scratch paper and make a clear note of it.

Question	Points	Score
1	10	
2	10	
3	10	
4	15	
5	50	
6	15	
Total:	110	

1. (10 points) True/False. Circle your answer. You do not need to show work.

(a) True False If f and g are 1-1 functions and and the composition $f \circ g$ makes sense, then $f \circ g$ is 1-1.

(b) True False $\pi^{\sqrt{2}} = \ln^{-1}(\sqrt{2}\ln(\pi))$.

(c) True False $\sin x$ is invertible on the interval $[0, \pi]$.

(d) True False $d/dx(e^{\pi}) = e^{\pi}$.

(e) True False $\ln(3^{20}) > 20$.

(f) True False The integral $\int_1^{\infty} x^{-0.2}dx$ converges.

(g) True False The rational function $\frac{2x^2+5}{(x^2+1)x^2}$ can be expressed in the form

$$\frac{2x^2+5}{(x^2+1)x^2} = \frac{A}{x^2+1} + \frac{B}{x} + \frac{C}{x^2}.$$

(h) True False Since $\int_0^{\pi/2} \sin^3 x dx = \frac{2}{3}$, then

$$\int_{-\pi/2}^{\pi/2} \sin^3 x dx = 2\int_0^{\pi/2} \sin^3 x dx = 2 \cdot \frac{2}{3} = \frac{4}{3}.$$

(i) True False If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = 3$, then

$$\lim_{x\to a} f(x)^{g(x)} = 1.$$

(j) True False If e^{-x} dominates $g(x)$ on the interval $[1, \infty)$, then $\int_1^{\infty} g(x)dx$ converges.

2. Inverses.

(a) (2 points) Show that $\varphi(x) = -\int_x^1 \cos^2 t + 1dt$ is invertible. [*hint*: besides using the definition of invertibility and the horizontal line test, what other criterion says that a function is invertible? We did this in the homework.]

$$\varphi(x) = \int_1^x \cos^2 t + 1\, dt$$

FTC I says

$$\frac{d\varphi}{dt} = \cos^2 x + 1 \quad \text{is positive for all } x \in \mathbb{R}$$

$\Longrightarrow \varphi$ is increasing

$\Longrightarrow \varphi$ is 1–1.

(b) (4 points) Show that $f(x) = \int_1^{1/x} \frac{1}{t}dt$ and $g(x) = e^{-x}$ are inverses of each other for $x > 0$.

$$f(x) = \ln(1/x) = -\ln x$$

$$f(g(x)) = f(e^{-x}) = -\ln(e^{-x}) = \ln e^{x} = x$$

$$g(f(x)) = g(-\ln x) = e^{-(-\ln x)} = x$$

(c) (4 points) Let $f(x) = \tan^{-1}(1/x)$ for $x \neq 0$. Compute $\frac{df^{-1}}{dx}$ at the point $x = f(1)$.

$$f'(x) = \frac{1}{(1/x)^2+1}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}$$

$$(f^{-1})'(f(1)) = \frac{1}{f'(f^{-1}(f(1)))} = \frac{1}{f'(1)} = -\frac{1}{1/2}$$

$$= \cancel{} -2$$

3. Limits. Compute the following limits.

(a) (5 points) $\lim_{x \to 0^+} x \cot x$ $\quad 0 \cdot \infty$

$$= \lim_{x \to 0^+} \frac{x}{\tan x} \qquad \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0^+} \frac{1}{\sec^2 x}$$

$$= \lim_{x \to 0^+} \cos^2 x$$

$$= 1$$

(b) (5 points) $\lim_{x \to 0} (1 + \sin x)^{2/x}$

Let $f(x) = (1+\sin x)^{2/x}$. Then

$$\lim_{x \to 0} \ln f(x) = \lim_{x \to 0} \frac{2 \ln(1+\sin x)}{x} \qquad \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0} 2 \frac{\frac{1}{1+\sin x} \cdot \cos x}{1}$$

$$= \lim_{x \to 0} \frac{2\cos x}{1+\sin x}$$

$$= \frac{2 \cdot 1}{1+0} = 2$$

$$\therefore \lim_{x \to 0} f(x) = e^2$$

5. Integration. Evaluate the following integrals

(a) (10 points) $\int_0^{1/2} \sqrt{1-4x^2}dx$ $= \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$

$u = 2x$

$du = 2dx$

$u = \sin\theta$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta}$

$= \cos\theta$

$du = \cos\theta\, d\theta$

$0 = \sin\theta \iff \theta = 0$

$1 = \sin\theta \iff \theta = \pi/2$

$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta$

$= \frac{1}{4}\int_0^{\pi/2} 1 + \cos 2\theta\, d\theta$

$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$

$= \frac{1}{4}\left(\left[\pi/2 + 0\right] - \left[0 + \frac{1}{2}\cdot 0\right]\right)$

$= \frac{\pi}{8}$

(b) (10 points) $\int \cot^4 x dx$

$\int \cot^4 x\, dx = \int (\csc^2 x - 1)\cot^2 x\, dx$

$= \int \csc^2 x \cot^2 x - \int \cot^2 x\, dx$

$u = \cot x$

$du = \csc^2 x\, dx$

$= \int \csc^2 x \cot^2 x - \int (\csc^2 x - 1)\, dx$

$= \int \csc^2 x \cot^2 x\, dx - \int \csc^2 x + \int 1\, dx$

$= -\frac{\cot^3 x}{3} - \cot x + x + C$

4. Derivatives. Find y'.

(a) (5 points) $y = (x^2+3x)(x-2)\sqrt{\ln x + 3}$

$$\ln y = \ln(x^2+3x) + \ln(x-2) + \frac{1}{2}\ln(\ln x + 3)$$

$$\frac{y'}{y} = \frac{1}{x^2+3x}(2x+3) + \frac{1}{x-2} + \frac{1}{2}\cdot\frac{1}{\ln x+3}\cdot\frac{1}{x}$$

$$y' = y\left(\frac{2x+3}{x^2+3x} + \frac{1}{x-2} + \frac{1}{2x(\ln x+3)}\right)$$

$$= (x^2+3x)(x-2)(\sqrt{\ln x+3})\left(\frac{2x+3}{x^2+3x} + \frac{1}{x-2} + \frac{1}{2x(\ln x+3)}\right)$$

(b) (5 points) $y = e^{1/x^2} + 1/e^{x^2}$

$$y' = e^{1/x^2}\cdot\frac{d}{dx}\left(\frac{1}{x^2}\right) + e^{-x^2}\cdot\frac{d}{dx}(-x^2)$$

$$= e^{1/x^2}\cdot\left(\frac{-2}{x^3}\right) + e^{-x^2}(-2x)$$

(c) (5 points) $y = \csc^{-1}(x^x) + \sec^{-1}(x^x)$

$$y' = -\frac{d}{dx}\csc^{-1}(x^x) + \frac{d}{dx}\sec^{-1}(x^x)$$

$$= 0$$

(c) (10 points) $\int_{-\infty}^{\infty} \frac{dy}{9+y^2} = 2\int_0^{\infty} \frac{dy}{9+y^2}$

$$= \lim_{b\to\infty} 2\int_0^b \frac{dy}{9+y^2}$$

$$= \lim_{b\to\infty} 2\left[\frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)\right]_0^b$$

$$= \frac{2}{3} \lim_{b\to\infty} \tan^{-1}\left(\frac{b}{3}\right)$$

$$= \frac{2}{3}\cdot\frac{\pi}{2}$$

$$= \pi/3$$

(d) (10 points) $\int x\cos^2 x \sin x dx = I$

Note $\int \cos^2 x \sin x\, dx = -\int u^2 du = -\frac{1}{3}u^3 = -\frac{1}{3}\cos^3 x$

$u = x \qquad dv = \cos^2 x \sin x\, dx$

$du = dx \qquad v = -\frac{1}{3}\cos^2 x$

$$I = -\frac{x}{3}\cos^2 x + \int \frac{1}{3}\cos^2 x\, dx$$

$$= -\frac{x}{3}\cos^2 x + \frac{1}{6}\int (1+\cos 2x)dx$$

$$= -\frac{x}{3}\cos^2 x + \frac{1}{6}\left(x + \frac{1}{2}\sin 2x\right) + C$$

(e) (10 points) $\int \frac{2x+21}{2x^2+9x-5}dx$

$$2x^2+9x-5 = 2x^2+10x-x-5$$
$$= 2x(x+5)-1(x+5)$$
$$= (2x-1)(x+5)$$

$$\frac{2x+21}{(2x-1)(x+5)} = \frac{A}{2x-1} + \frac{B}{x+5}$$

$$2x+21 = Ax+5A+2Bx-B$$
$$= (A+2B)x + (5A-B)$$

$$\begin{cases} A+2B = 2 \\ 5A-B = 21 \end{cases} \rightarrow \begin{cases} A+2B = 2 \\ 10A-2B = 21 \end{cases} \rightarrow \begin{cases} 11A = 23 \\ 10A-2B = 21 \end{cases}$$

$$\rightarrow A = 23/11 \rightarrow \frac{23}{11} + 2B = 2$$
$$2B = \frac{22}{11} - \frac{23}{11}$$
$$= -\frac{1}{11}$$
$$B = -\frac{1}{22}$$

$$\int \frac{2x+21}{2x^2+9x-5}dx = \frac{23}{11}\int \frac{dx}{2x-1} - \frac{1}{22}\int \frac{dx}{x+5}$$

$$= \frac{23}{11}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{22}\ln|x+5| + C.$$

6. Differential equations and exponential change.

(a) (4 points) Show that $y = x \ln x + x$ is a solution to the differential equation $y' - x^{-1}y = 1$.

$$y' = \left(x \cdot \frac{1}{x} + \ln x\right) + 1 = 2 + \ln x$$

$$\frac{y}{x} = \frac{x \ln x + x}{x} = \ln x + 1$$

$$y' - \frac{y}{x} = 2 + \cancel{\ln x} - \cancel{\ln x} - 1$$

$$= 1.$$

✓

(b) (6 points) Solve the differential equation $e^{y'} = x^{y+1}$.

$$\ln e^{y'} = \ln x^{y+1}$$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x} \qquad v = x$$

$$y' = (y+1)\ln x$$

$$\int \frac{dy}{y+1} = \int \ln x \, dx$$

$$\ln|y+1| = x \ln x - x + C$$

$$|y+1| = e^{x \ln x - x + C}$$

$$y+1 = A e^{x \ln x} e^{-x} \qquad A = \pm e^{C}$$

$$y = A x^{x} e^{-x} \cancel{B} - 1$$

(c) (5 points) The rate of growth of a colony of bacteria is proportional to the population's size at any given time. At the end of 3 hours there are 1000 bacteria. At the end of 4 hours there are 4000 bacteria. How many bacteria were there initially?

$$y = y_0 e^{kt}$$

$$1000 = y(3) = y_0 e^{k \cdot 3}$$

$$\frac{1000}{y_0} = e^{k \cdot 3}$$

$$\ln\left(\frac{1000}{y_0}\right) = k \cdot 3$$

$$k = \frac{1}{3}\ln\left(\frac{1000}{y_0}\right)$$

$$4000 = y(4) = y_0 e^{k \cdot 4} = y_0 e^{\frac{1}{3}\ln\left(\frac{1000}{y_0}\right) \cdot 4}$$

$$\ln 4000 = \ln y_0 + \frac{1}{3}\ln\left(\frac{1000}{y_0}\right) \cdot 4$$

$$= \ln y_0 + \frac{4}{3}\ln(1000) - \frac{4}{3}\ln(y_0)$$

$$= -\frac{1}{3}\ln y_0 + \frac{4}{3}\ln(1000)$$

$$\ln 4000 - \frac{4}{3}\ln 1000 = -\frac{1}{3}\ln y_0$$

$$\ln y_0 = 4\ln 1000 - 3\ln 4000$$

$$= \ln \frac{1000^4}{4000^3}$$

$$y_0 = \frac{1000^4}{4000^3}$$