MATH 242 Summer 2016
Practice Exam 2

Name: Solutions #1

Instructions:

- Begin by writing your name in the space above.
- You have 80 minutes to complete this exam.
- No phones, calculators, notes, or any form of assistance may be used during the exam.
- You must show all of your work, unless you are asked not to. Answers which are cryptic or have no supporting evidence will most likely not receive full credit. When in doubt, ask.
- Please be organized! Answer questions in the space provided as neatly as possible. If you run out of room, continue on a piece of scratch paper and make a clear note of it.

Question	Points	Score
1	7	
2	10	
3	17	
4	20	
5	30	
6	22	
Total:	106	

1. (7 points) True/False. Circle your answer. You do not need to show work. Each correct answer is worth 1 point, an incorrect answer is worth -1.5 points. If you do not want to be marked on any problem write the symbol "Z" next to the problem; if you do this, then you will neither gain or lose points.

(a) True **False** If $\sum a_n$ is a convergent series with nonnegative terms, then $\sum(-1)^n a_n$ may diverge.

(b) True **False** If $\{a_n\}$ and $\{b_n\}$ both diverge, then $\{a_n + b_n\}$ must diverge.

(c) True **False** If the even terms in a sequence converge to 0 and the odd terms converge to 1, then the entire sequence converges to the average 1/2.

(d) True **False** Since $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, then the series $\sum_{n=1}^{\infty} \frac{1}{\ln n}$ converges.

(e) True **False** If $0 \le a_n \le b_n$ for all $n > N$ (N some integer) and the series $\sum b_n$ diverges, then the series $\sum a_n$ diverges.

(f) True **False** If $\sum a_n^2$ diverges, then $\sum a_n$ diverges.

(g) True **False** If $a_n \le b_n \le c_n$ and the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} c_n$ both exist, then the limit $\lim_{n\to\infty} b_n$ exists.

(a) $\sum |(-1)^n a_n| = \sum a_n$ converges so $\sum (-1)^n a_n$ converges absolutely.

(b) $\{a_n\} = \{n\}$, $\{b_n\} = \{-n\}$ both diverge, but $\{a_n + b_n\} = \{0\}_{n=1}^{\infty}$ converges.

(c) nonsense.

(d) $\sum_{n=1}^{\infty} \frac{1}{\ln n}$ diverges DCT with the harmonic

(e) $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. The series $\sum b_n$ diverges but $\sum a_n$ converges

(f) $a_n = (-1)^n \frac{1}{\sqrt{n}}$. The series $\sum a_n^2 = \sum \frac{1}{n}$ diverges but $\sum a_n$ converges by AST

(g) $a_n = -1$, $c_n = 1$ and $b_n = \sin n$ is a counterexample

2. Sequences.

(a) (2 points) State the monotone sequence theorem.

Bounded, monotone sequences converge.

(b) (3 points) Give an example of sequence which is *not* monotone and converges to $-1/2$.

Let $a_n = \frac{\sin n}{n} - \frac{1}{2}$. Then

$$-\frac{1}{n} - \frac{1}{2} \leq \frac{\sin n}{n} - \frac{1}{2} \leq \frac{1}{n} - \frac{1}{2}$$

Let $n \to \infty$

$$-\frac{1}{2} \leq \lim_{n\to+\infty} \left(\frac{\sin n}{n} - \frac{1}{2}\right) \leq -\frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{\sin n}{n} - \frac{1}{2}\right) = -\frac{1}{2}$$

(c) (5 points) Find the limit of the sequence $\{|\sin n|(\ln n)^{-1}\}_{n=1}^{\infty}$

$$\frac{0}{\ln n} \leq \frac{|\sin n|}{\ln n} \leq \frac{1}{\ln n}$$

Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, then by Sandwich thm $\lim_{n\to\infty} \frac{|\sin n|}{\ln n} = 0$.

3. Series.

(a) (2 points) State the bounded sum test.

A series with nonnegative terms converges if and only if the sequence of partial sums is bounded.

(b) (5 points) Find $\sum a_n$ if its n-th partial sum is $1+\frac{1}{n(n+1)}$.

$$S_n = 1 + \frac{1}{n(n+1)}$$

$$\Longrightarrow \sum a_n = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n(n+1)}\right)$$

$$=1.$$

(c) (5 points) Find $\sum_{n=1}^{\infty} 4^{1-n}\pi^n$.

$$\sum_{n=1}^{\infty} 4^{1-n}\pi^n = \sum_{n=1}^{\infty} \pi\left(\frac{\pi}{4}\right)^{n-1} = \frac{\pi}{1-\pi/4} = \frac{4\pi}{4-\pi}$$

(d) (5 points) Find $\sum_{n=2}^{\infty} \frac{-2}{n^2-1}$.

$$\frac{-2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1}$$

$$\Longrightarrow \quad -2 = (A+B)n + (A-B)$$

$$\begin{cases} A+B = 0 \\ A-B = -2 \end{cases} \longrightarrow \begin{cases} A+B = 0 \\ 2A = -2 \end{cases} \longrightarrow \begin{cases} A+B = 0 \\ A = -1 \end{cases} \Longrightarrow B = 1$$

$$\begin{aligned} S_k &= \left(-\frac{1}{1} + \cancel{\frac{1}{3}}\right) + \left(-\frac{1}{2} + \cancel{\frac{1}{4}}\right) \\ &+ \left(-\cancel{\frac{1}{3}} + \cancel{\frac{1}{5}}\right) + \left(-\cancel{\frac{1}{4}} + \cancel{\frac{1}{6}}\right) \\ &+ \cdots \\ &+ \left(-\cancel{\frac{1}{k-2}} + \frac{1}{k}\right) + \left(-\cancel{\frac{1}{k-1}} + \frac{1}{k+1}\right) = -\frac{3}{2} + \frac{1}{k} + \frac{1}{k+1} \end{aligned}$$

$$\therefore \sum_{n=2}^{\infty} \frac{-2}{n^2-1} = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(-\frac{3}{2} + \frac{1}{k} + \frac{1}{k+1}\right) = -3/2$$

4. Series Continued. Determine whether the following series converge or diverge.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+4)!}{3!(n+1)!\,3^{n+1}} \cdot \frac{3!\,n!\,3^n}{(n+3)!} = \frac{(n+4)!}{(n+3)!}\,\frac{3!}{3!}\,\frac{n!}{(n+1)!}\,\frac{3^n}{3^{n+1}}$$

$$= \frac{n+4}{n+1} \cdot \frac{1}{3}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+4}{n+1} \cdot \frac{1}{3} \stackrel{H}{=} \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1$$

$\therefore$ the series converges by the ratio test.

(b) (5 points) $\sum_{n=1}^{\infty} n\sin(1/n)$

$$\lim_{n\to\infty} n\sin(1/n) \qquad \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \qquad \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{n\to\infty} \frac{\cos(1/n)\cdot(-1/n^2)}{-1/n^2}$$

$$= \lim_{n\to\infty} \cos(1/n) = 1 \neq 0$$

$\therefore$ The series diverges by the n^{th} term test.

(c) (5 points) $\sum_{n=1}^{\infty}(-1)^n(n^{-2}+2^{-n})$

$$\sum_{n=1}^{\infty}\left|(-1)^n(n^{-2}+2^{-n})\right| = \sum_{n=1}^{\infty}(n^{-2}+2^{-n}) = \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$$

p-series $p=2>1$

geometric series with $|r| = \left|\frac{1}{2}\right| < 1$

$\therefore$ the series converges by the ACT.

(d) (5 points) $\sum_{n=1}^{\infty}\frac{3n^2+1}{n^3-4}$

Let $a_n = \frac{3n^2+1}{n^3-4}$, pick $b_n = \frac{3n^2}{n^3} = \frac{3}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{3n^2+1}{n^3-4}\cdot\frac{n}{3} = \lim_{n\to\infty}\frac{3n^3+n}{n^3-4}\cdot\frac{1}{3}$$

$$= \lim_{n\to\infty}\frac{3+1/n^2}{1-4/n^3}\cdot\frac{1}{3}$$

$$= 1$$

Since the limit is nonzero and finite and $\sum b_n$ diverges then $\sum a_n$ diverges by the LCT.

5. Power series.

(a) (10 points) Find a power series representation for $\dfrac{e^x + e^{-x}}{2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x + e^{-x} = 2 + 2 \cdot \frac{x^2}{2!} + 2\,\frac{x^4}{4!} + \cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

(b) (10 points) Use power series to compute the limit $\displaystyle\lim_{x\to\infty} \frac{(\sin x)/x - \cos x}{x^2}$.

$$\frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$\frac{\sin x}{x} - \cos x = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= \left(\frac{1}{2!} - \frac{1}{3!}\right)x^2 + \left(\frac{1}{5!} - \frac{1}{4!}\right)x^4 + \cdots$$

$$\lim_{x\to\infty} \frac{\sin x/x - \cos x}{x^2} = \lim_{x\to\infty}\left(\left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{5!} - \frac{1}{4!}\right)x^2 + \cdots\right)$$

$$= \infty$$

(c) (10 points) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3x-1)^{n+1}}{n+1} \cdot \frac{n}{(3x-1)^n} \right|$$

$$= \lim_{n\to\infty} |3x-1| \frac{n}{n+1}$$

$$= |3x-1|$$

By the ratio test the series converges for

$$|3x-1| < 1 \iff -1 < 3x-1 < 1$$

$$\iff 0 < 3x < 2$$

$$\iff 0 < x < 2/3$$

$x=0$: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ is the alternating harmonic so converges.

$x = 2/3$: $\sum_{n=1}^{\infty} \frac{(3 \cdot 2/3 - 1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic so diverges.

$\therefore$ The interval of convergence of the series is

$$0 \le x < 2/3$$

6. Taylor series.

(a) (2 points) State Taylor's theroem.

If f and its first n derivatives are continuous on the closed interval between a and b, and $f^{(n+1)}$ is differentiable on the open interval between a and b, then $\exists c$ between a and b s.t.

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

(b) (10 points) Find the Taylor series generated by $f(x) = \ln x$ at $x = 2$.

$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2}$

$f'''(x) = \frac{1}{x^3} \quad f^{(4)}(x) = -\frac{1}{x^4}$

$f^{(n)}(x) = (-1)^{n+1}\frac{1}{x^n}$ for $n \geq 1$

$\Rightarrow f^{(n)}(2) = (-1)^{n+1}\frac{1}{2^n}$

The Taylor series is

$$\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot\frac{1}{2^n}}{n!}(x-2)^n$$

$$= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!\cdot 2^n}(x-2)^n$$

(c) (10 points) Find the Maclaurin series generated by $f(x) = e^{2x}$.

$f(x) = e^{2x} \quad f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x} \quad f'''(x) = 2^3 e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

So $f^{(n)}(0) = 2^n$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$