MATH 242 Summer 2016
Practice Exam 2

Name: Solutions #2

Instructions:

- Begin by writing your name in the space above.
- You have 80 minutes to complete this exam.
- No phones, calculators, notes, or any form of assistance may be used during the exam.
- You must show all of your work, unless you are asked not to. Answers which are cryptic or have no supporting evidence will most likely not receive full credit. When in doubt, ask.
- Please be organized! Answer questions in the space provided as neatly as possible. If you run out of room, continue on a piece of scratch paper and make a clear note of it.

Question	Points	Score
1	7	
2	10	
3	17	
4	20	
5	30	
6	22	
Total:	106	

1. (7 points) True/False. Circle your answer. You do not need to show work. Each correct answer is worth 1 point, an incorrect answer is worth −1.5 points. If you do not want to be marked on any problem write the symbol "Z" next to the problem; if you do this, then you will neither gain or lose points.

(a) True **False** Maclaurin series always converge to the generating function.

(b) **True** False $e^{-i\pi} = -1$.

(c) **True** False If the sequence of n-th partial sums of a series with nonnegative terms is bounded, then the series converges.

(d) True **False** If $\sum a_n$ diverges, then its sequence of n-th partial sums is unbounded.

(e) **True** False If $\sum a_n$ converges, then $a_n \to 0$.

(f) True **False** If $\{a_n\}$ diverges, then $\{a_n/n\}$ diverges.

(g) **True** False If $\{a_n\}$ is a sequence converging to a real number L, then

$$\lim_{n\to\infty} \cos(\ln(|a_n| + 1)) = \cos(\ln(|L| + 1)).$$

(h) True **False** A power series may converge at exactly two distinct real numbers.

(a) See example 4 in 9.8.

(b) $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1$

(c) Bounded Sum test. (Corollary of thm 6 in 9.3)

(d) The series $\sum_{n=1}^{\infty}(-1)^n$ diverges by the n^{th} term test, but $S_n = \begin{cases} -1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$

Is bounded.

(e) N^{th} term test stated differently

(f) $\{a_n\} = \{n\}$ diverges but $\{a_n/n\} = \{1\}$ converges to 1.

(g) cos, ln and the absolute value are continuous so the limit can be "pushed in"

(h) Power series always converge on an interval ~~isolated~~ $[a, a] = \{a\}$. points are degenerate intervals and are included

2. Sequences.

(a) (2 points) What does it mean when we say "f interpolates the sequence $\{a_n\}$"?

$$f(n) = a_n$$

(b) (3 points) Give an example of a monotone sequence that converges to -1.

$$a_n = \frac{1}{n} - 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n} - 1\right) = -1$$

(c) (5 points) Find the limit of the sequence $a_n = (e^{e^n})^{1/n!}$.

$$a_n = \left(e^{e^n}\right)^{1/n!} = e^{e^n/n!}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{e^n/n!}$$

$$= e^{\lim_{n\to\infty} e^n/n!} = e^0 = 1$$

Continuity of exponential function.

3. Series.

(a) (2 points) What is the formal definition of infinite series.

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k \quad \text{where } S_k = a_1 + a_2 + \cdots + a_k$$

(b) (5 points) Sow that $0.\overline{2} = 2/9$ using geometric series.

$$0.\overline{2} = \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{2}{10^n}$$

$$= \sum_{n=1}^{\infty} \frac{2}{10}\cdot\left(\frac{1}{10}\right)^n = \frac{2/10}{1-1/10}$$

$$= \frac{2}{10-1} = \frac{2}{9}$$

4. Series.

(a) (5 points) Use the integral test to determine whether $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$ converges or diverges.

Let $f(x) = \frac{1}{4x^2+1}$

$$\int f(x)\,dx = \int \frac{dx}{4x^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u = \frac{1}{2}\tan^{-1} 2x$$

$$\therefore \int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \frac{1}{2}\tan^{-1} 2x \Big|_1^b = \lim_{b\to\infty} \frac{1}{2}\tan^{-1} 2b - \frac{1}{2}\tan^{-1} 2$$

$$= \pi/4 - \frac{1}{2}\tan^{-1} 2 < \infty$$

$\therefore$ Since $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$ converges

(b) (5 points) Classify $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+\sqrt{n}}$ as absolutely convergent, conditionally convergent, or divergent.

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+\sqrt{n}}$

- $\frac{1}{n+\sqrt{n}} > 0$ for all n

- $\frac{1}{n+1+\sqrt{n+1}} < \frac{1}{n+\sqrt{n}}$ for all n

- $\frac{1}{n+\sqrt{n}} \to 0$

$\therefore \sum a_n$ converges by AST

$\sum |a_n| = \sum \frac{1}{n+\sqrt{n}}$. Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt{n}} = 1$$

$\therefore \sum |a_n|$ diverges by limit comparison w/ harmonic

$\sum a_n$ is conditionally convergent

(c) (5 points) Find $\sum a_n$ if its n-th partial sum is $\frac{2^n - 1}{2^n - 2^{n-1}}$.

$$S_n = \frac{2^n - 1}{2^n - 2^{n-1}} = \frac{1 - 1/2^n}{1 - 1/2}$$

$$\sum a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - 1/2^n}{1 - 1/2}$$

$$= \frac{1}{1 - 1/2}$$

$$= 2$$

(d) (5 points) Find $\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$.

$$S_k = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cdots + \left(\cancel{\frac{1}{2k-1}} - \frac{1}{2k+1}\right)$$

$$= 1 - \frac{1}{2k+1}$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = \lim_{k\to\infty} S_k = 1 .$$

(c) (5 points) Does the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ converge or diverge?

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\therefore$ the series diverges by the

n^{th} term test.

(d) (5 points) Does the series $\sum_{n=1}^{\infty}\left(\frac{1}{\ln n}\right)^n$ converge or diverge?

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$$

$\therefore$ the series converges by

the root test.

5. Power series.

(a) (10 points) Find a power series representation for $\tan^{-1}(x)/x$.

$$\tan^{-1}(x) = 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\frac{\tan^{-1}x}{x} = x^{-1} - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots$$

(b) (10 points) Express $\int xe^x dx$ as a power series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$$

$$\int xe^x = \int \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots \right) dx$$

$$= \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4 \cdot 2!} + \frac{x^5}{5 \cdot 3!} + \cdots$$

6. Taylor series.

(a) (2 points) What is Taylors formula?

$$f(x) = P_n(x) + R_n(x)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

(b) (10 points) Find the Taylor polynomial of order 3 generated by $f(x) = x \sin x$ at $x = \pi/2$.

$f'(x) = \sin x + x\cos x$

$f''(x) = \cos x + \cos x - x\sin x$

$= 2\cos x - x\sin x$

$f'''(x) = -2\sin x - \sin x - x\cos x$

$= -3\sin x - x\cos x$

$f'(\pi/2) = 1$

$f''(\pi/2) = -\pi/2$

$f'''(\pi/2) = -3$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= \frac{\pi}{2} + 1\cdot(x - \pi/2) + \frac{-\pi/2}{2!}(x-\pi/2)^2 + \frac{-3}{3!}(x-\pi/2)^3$$

(c) (10 points) Find the Maclaurin series generated by $f(x) = 2^x$.

$f(x) = 2^x \quad f'(x) = 2^x \ln 2$

$f''(x) = 2^x(\ln 2)^2 \quad f'''(x) = 2^x(\ln 2)^3$

$\vdots$

$f^{(n)}(x) = 2^x(\ln 2)^n$

$f^{(n)}(0) = 2^0(\ln 2)^n = (\ln 2)^n$

The taylor series is then

$$\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n$$

(c) (10 points) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n} \right| = 3 \cdot \frac{1}{n+1} \cdot |x|$$

$$\Longrightarrow \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3}{n+1} \cdot |x| = 0$$

$\therefore$ the series converges absolutely for all x.

The interval of convergence is $(-\infty, \infty)$.