MATH 242 Summer 2016
Practice Exam 2

Name: Solutions. #3

Instructions:

- Begin by writing your name in the space above.
- You have 80 minutes to complete this exam.
- No phones, calculators, notes, or any form of assistance may be used during the exam.
- You must show all of your work, unless you are asked not to. Answers which are cryptic or have no supporting evidence will most likely not receive full credit. When in doubt, ask.
- Please be organized! Answer questions in the space provided as neatly as possible. If you run out of room, continue on a piece of scratch paper and make a clear note of it.

Question	Points	Score
1	8	
2	10	
3	17	
4	20	
5	15	
6	12	
Total:	82	

1. (8 points) True/False. Circle your answer. You do not need to show work. Each correct answer is worth 1 point, an incorrect answer is worth −1.5 points. If you do not want to be marked on any problem write the symbol "Z" next to the problem; if you do this, then you will neither gain or lose points.

   (a) True False Every function generates a Maclaurin series.

   (b) True False A p-series converges when $p \geq 1$ and diverges when $0 < p < 1$.

   (c) True False Suppose $0 \leq a_n \leq b_n$ and $\sum a_n$ converges, then $\sum b_n$ converges.

   (d) True False The Maclaurin series generated by a polynomial is the polynomial itself.

   (e) True False An alternating p-series converges absolutely for $p \geq 1$ and converges conditionally for $0 < p \leq 1$.

   (f) True False If $a_n \leq b_n \leq c_n$ and both a_n and c_n converge to 0, then $\lim_{n\to\infty} e^{b_n} = 1$.

   (g) True False A power series may converge everywhere except at one number.

   (h) True False The complex exponential is not 1-1 since $e^{-i\pi} = -1$ and $e^{i\pi} = -1$.

(a) the function needs to be infinitely differentiable
Not <u>every</u> function is.

(b) ~~$p \geq 1$ when~~ Converges when $p > 1$ and diverges when $0 < p \leq 1$.

(c) $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$

(d) Exercise: show that the Maclaurin series of a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is $p(x)$.

(e) Converge absolutely for $p > 1$ and conditionally for $0 < p \leq 1$

(f) Sandwich theorem says $b_n \to 0$ so
$$\lim_{n\to\infty} e^{b_n} = e^{\lim_{n\to\infty} b_n} = e^0 = 1.$$

(g) Power series <u>always</u> converge on an interval. $(-\infty, a) \cup (a, \infty)$ is not an interval.

(h) $e^{-i\pi} = e^{i\pi}$ but $-i\pi \neq i\pi$ so not 1-1.

2. Sequences.

(a) (2 points) State the sandwich theorem.

Suppose $a_n \le b_n \le c_n$ and that both a_n and c_n converge to L, then $b_n \to L$.

(b) (3 points) Give an example of how the sandwich theorem is applied.

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \quad \text{for all } n$$

Since both $-1/n$ and $1/n$ converge to 0,

then $\frac{\sin n}{n} \to 0$.

(c) (5 points) Find the limit of the sequence $a_n = \dfrac{n + \ln n}{n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+\ln n}{n} = \lim_{n\to\infty}\left(1 + \frac{\ln n}{n}\right)$$

$$= 1 + \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\overset{H}{=} 1 + \lim_{n\to\infty} \frac{1/n}{1} = 1.$$

3. Series.

(a) (2 points) State the root test.

Let ~~XXXX~~ $\sum a_n$ be a series s.t. $a_n > 0$ for all $n > N$ (N some integer). Let $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \rho$

(1) If $\rho < 1$, then $\sum a_n$ converges

(2) If $\rho > 1$, then $\sum a_n$ diverges

(3) If $\rho = 1$, the test is inconclusive.

(b) (5 points) If a series has n-th partial sum $s_n = \dfrac{1-(0.9)^n}{1-(0.9)}$, what is its sum?

$$S_n = \frac{1-(0.9)^n}{1-(0.9)}$$

$$\sum a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-(0.9)^n}{1-(0.9)}$$

$$= \frac{1}{1-0.9}$$

$$= \frac{1}{1/10} = 10$$

(c) (5 points) Find $\sum_{n=2}^{\infty} \frac{-2}{n^2+n}$.

$$\frac{-2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$-2 = (A+B)n + A$$

$$\begin{cases} A+B=0 \\ A \quad = -2 \end{cases} \Longrightarrow B = 2$$

$$S_k = \left(\frac{-2}{2} + \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{-2}{3}} + \cancel{\frac{2}{4}}\right) + \left(\cancel{\frac{-2}{4}} + \cancel{\frac{2}{5}}\right) + \cdots + \left(\cancel{\frac{-2}{k}} + \frac{2}{k+1}\right)$$

$$= -1 + \frac{2}{k+1}$$

$$\therefore \sum_{n=2}^{\infty} \frac{-2}{n^2+n} = \lim_{k\to\infty}\left(-1 + \frac{2}{k+1}\right) = -1$$

(d) (5 points) Find $\sum_{n=2}^{\infty} \frac{10!}{2^n}$.

$$\sum_{n=2}^{\infty} \frac{10!}{2^n} = \sum_{n=1}^{\infty} \frac{10!}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{10!}{2^2} \cdot \left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{10!/4}{1-1/2} = \frac{10!}{2}$$

4. Series.

(a) (5 points) Use the integral test to show that $\sum_{n=1}^{\infty} e^{-n}$ converges.

Let $f(x) = \frac{1}{e^x}$ (positive, decreasing, cont, interpolates terms)

$$\int \frac{1}{e^x}\,dx = -e^{-x}$$

$$\Rightarrow \int_1^{\infty} \frac{1}{e^x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{e^b} + \frac{1}{e} = \frac{1}{e} < \infty$$

Since $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} e^{-n}$ converges

(b) (5 points) Classify $\sum_{n=1}^{\infty} \frac{\cos n}{n\sqrt{n}}$ as absolutely convergent, conditionally convergent, or divergent.

Note $|\cos n| \leq 1$ for all n. So that

$$\left|\frac{\cos n}{n\sqrt{n}}\right| \leq \frac{1}{n\sqrt{n}} \text{ for all } n.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (P-Series $p = 3/2 > 1$)

then $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n\sqrt{n}}\right|$ converges.

Hence, $\sum_{n=1}^{\infty} \frac{\cos n}{n\sqrt{n}}$ converges by ACT.

(c) (5 points) What does the ratio test say about p-series? $a_n = \frac{1}{n^p}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p$$

$$= \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^p \qquad f(x) = x^p \text{ is continuous}$$

$$= 1^p$$

$$= 1$$

$\therefore$ Ratio test is inconclusive.

(d) (5 points) Does the series $\sum_{n=1}^{\infty} \sin(1/n)$ converge or diverge?

Let $a_n = \sin(1/n)$ and $b_n = 1/n$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \qquad \frac{0}{0}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{\cos(1/n)\cdot(-1/n^2)}{(-1/n^2)}$$

$$= \lim_{n\to\infty} \cos(1/n) = 1$$

Since the limit is finite and nonzero then

$\sum a_n$ diverges as $\sum b_n$ diverges (LCT)

5. Power series.

(a) (5 points) Find a power series representation for $\cos^2 x$ [*hint*: use the half-angle formula].

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$= \frac{1 + \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)}{2}$$

$$= 1 - 2\frac{x^2}{2!} + 2\frac{x^4}{4!} - 2\frac{x^6}{6!} + \cdots$$

(b) (5 points) Show that $y = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$ is a solution to the differential equation $y' - 2y = 0$.

$$y = 1 + 2x + \frac{2^2 x^2}{2!} + \frac{2^3 x^3}{3!} + \frac{2^4 x^4}{4!} + \cdots$$

$$y = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

$$y' = 0 + 2 + \frac{2(2x)^1}{2!}\cdot 2 + \frac{3(2x)^2}{3!} + \frac{4(2x)^3}{4!}\cdot 2 + \cdots$$

$$= 2\left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right)$$

$$= 2y$$

$$\therefore\ y' - 2y = 0.$$

(c) (5 points) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}\,x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}\,x^n}\right| = \sqrt{\frac{n+1}{n}}\,|x| \cdot \frac{1}{3}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}\;\frac{|x|}{3}\right)$$

$$= \frac{|x|}{3}\sqrt{\lim_{n\to\infty}\frac{n+1}{n}} \qquad \left(\text{Square root is continuous}\right)$$

$$= \frac{|x|}{3}\cdot 1$$

Power Series converges absolutely for $\frac{|x|}{3} < 1$

so $|x| < 3 \iff -3 < x < 3$

check endpoints for conditional convergence.

$x=-3$: $\sum_{n=0}^{\infty} \frac{\sqrt{n}(-1)^n 3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n\sqrt{n}$ diverges by n^{th} term test.

$x=3$: $\sum_{n=0}^{\infty} \frac{\sqrt{n}\,3^n}{3^n} = \sum_{n=0}^{\infty} \sqrt{n}$ diverges by n^{th} term test.

$\therefore$ Interval of convergence is $-3 < x < 3$.

6. Taylor series.

(a) (2 points) What is a Maclaurin series?

The Taylor Series generated by a function at $x=0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

(b) (5 points) Find the Taylor series generated by $f(x) = \cos x$ at $x = \pi/2$.

$f(x) = \cos x \qquad f'(x) = -\sin x$

$f''(x) = -\cos x \qquad f'''(x) = \sin x$

$f^{(4)}(x) = \cos x \qquad f^{(5)}(x) = -\sin x$

~~$f^{(n)}(x) = (-1)^n$~~

$f^{(2k)}(x) = (-1)^k \cos x \qquad f^{(2k+1)}(x) = (-1)^{k+1} \sin x$

$f^{(2k)}(\pi/2) = 0 \qquad f^{(2k+1)}(\pi/2) = (-1)^{k+1}$

Taylor series

$$f(\pi/2) + f'(\pi/2)(x-\pi/2) + \frac{f''(\pi/2)}{2!}(x-\pi/2)^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\pi/2)^3 + \cdots$$

$$= 0 - (x-\pi/2) + 0 + \frac{(x-\pi/2)^3}{3!} + 0 + \cdots$$

$$= -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(x-\pi/2)^{2k+1}$$

(c) (5 points) Find the Maclaurin series generated by $f(x) = \dfrac{1}{1-x}$.

$f(x) = \dfrac{1}{1-x}$

$f'(x) = \dfrac{1 \cdot 2}{(1-x)^2}$

$f''(x) = \dfrac{1 \cdot 2 \cdot 3}{(1-x)^3}$

$f^{(n)}(x) = \dfrac{1 \cdot 2 \cdot 3 \cdots n}{(1-x)^{n+1}}$

$f^{(n)}(0) = \dfrac{1}{(1-0)^{n+1}} = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{n!\,x^n}{n!}$$

$$= \sum_{n=0}^{\infty} x^n$$